DEVELOPING AND TESTING A LAGRANGIAN MODEL OF THE FLOATING-ARM TREBUCHET

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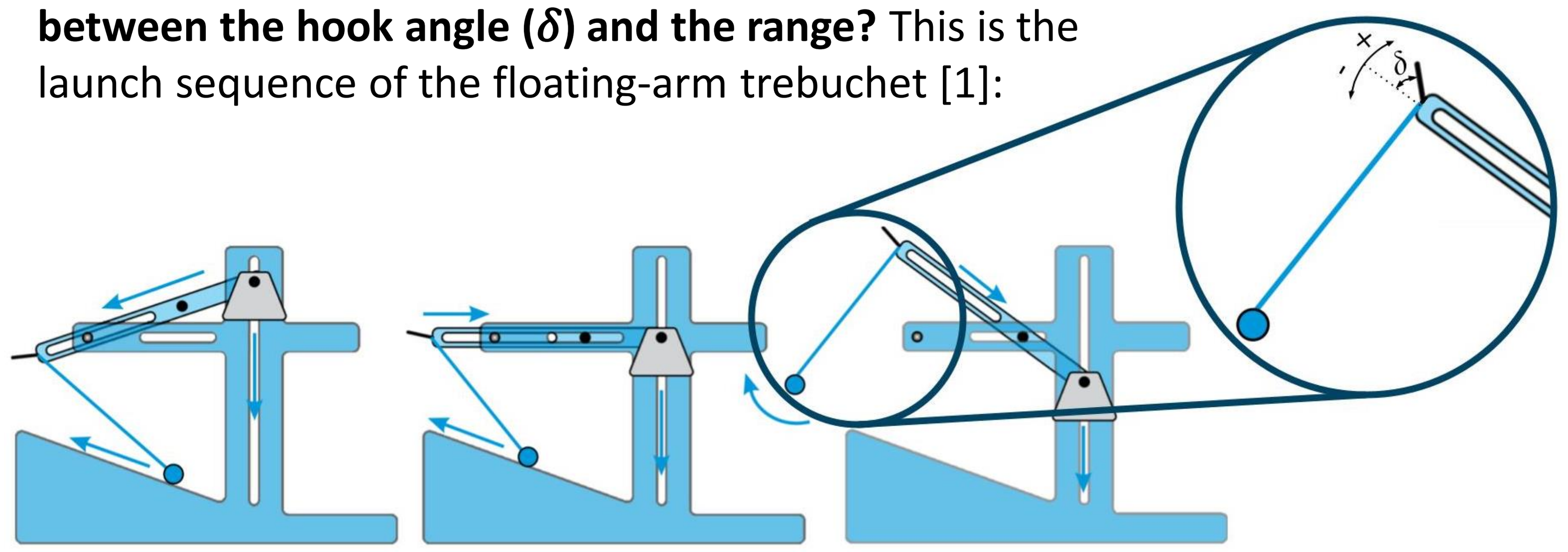
Introduction

This paper investigates the floating-arm trebuchet (a kind of catapult) which is a more energy-efficient modern variant of the medieval trebuchet. While the regular trebuchet is well described in literature due to its historical significance, the floating-arm variant has received little attention. Most notably, experimental verification of mathematical models is still lacking from the literature. Here, based on a Lagrangian model, I developed a predictive model and compared it to empirical results from a small-scale floating-arm trebuchet to address the following question : **What is the relationship between the hook angle (δ) and the range?** This is the launch sequence of the floating-arm trebuchet [1]:



1. Moving arm rests on fixed roller 2. Arm rests on moving roller 3. Projectile has left ramp

Methodology

Theoretical framework:

In order to develop a predictive model for the floating-arm trebuchet, let's have a look at the literature:

- To mathematically model the process illustrated above, Constans used Lagrangian mechanics [1]. Lagrangian mechanics is an alternative formulation of classical mechanics which is more appropriate than Newtonian mechanics for complex systems like the floating-arm trebuchet, due to its more fundamental approach [2]. By calculating the total kinetic energy of the system and the total potential energy of the system, we can find the “Lagrangian” ($\mathcal{L}$), a mathematical object that describes the floating-arm trebuchet in any configuration.

$$\mathcal{L} = \underbrace{\frac{1}{2}m_1\dot{h}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m_2\left(\dot{x_p}^2 + \dot{y_p}^2\right)}_{\text{kinetic energy}} \underbrace{- m_1gh - m_2gy_p}_{\text{potential energy}}$$

Skipping a lot of mathematical details, the behaviour of the floating-arm trebuchet can be predicted by iteratively solving a system of linear equations.

$$\begin{bmatrix} \boldsymbol{M} & \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\right)^T \\ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{g} \\ \boldsymbol{\gamma} \end{bmatrix}$$

- Additionally, Siano developed a launch condition for the regular trebuchet in terms of the hook angle (δ), which also applies here to the floating-arm trebuchet [3].

$$\alpha = \pi + \delta - \operatorname{arccot} \mu$$

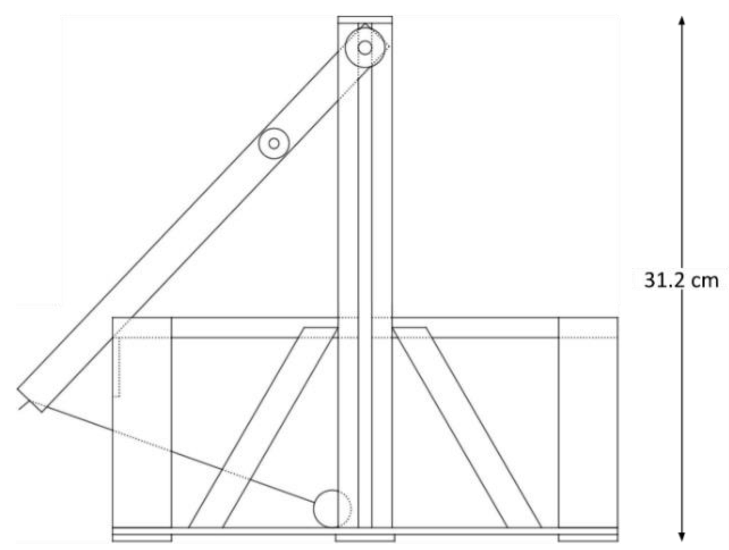
- Finally, Hill describes a numerical solution to the famous problem of projectile motion with air resistance [4].

Developing my prediction theoretical model

Computer simulation of Lagrangian model by Constans [1]
→ Launch condition based on hook angle by Siano [3]
→ Numerical analysis of projectile motion with air resistance by Hill [4]
→ **Predictive theoretical model for the relationship between the hook angle and the range of a floating-arm trebuchet**

Building my trebuchet

- A physical floating-arm trebuchet was built based on instructions by Akiyama [5].

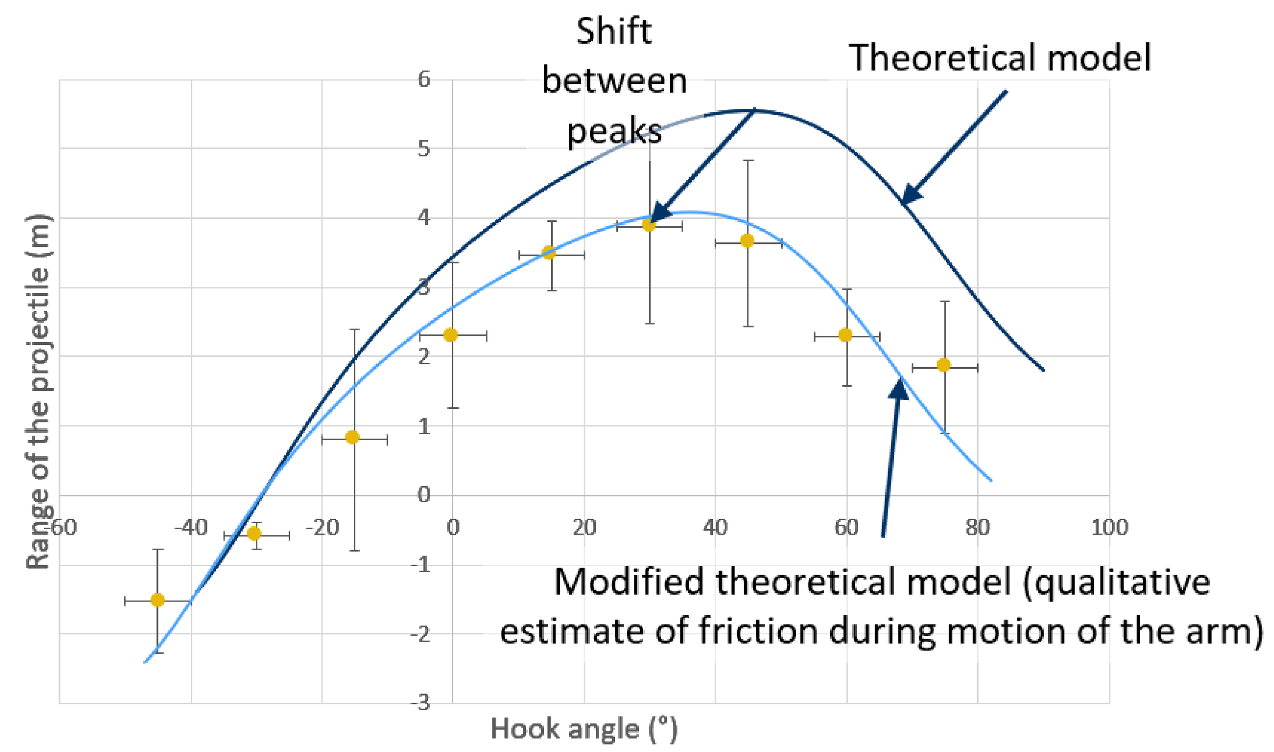


- Small scale model was able to throw marbles at a distance of 5 meters despite only being 30 centimetres tall and being powered by gravity alone.
- Dimensions were accurately measured to use as parameters in the simulations.
- The Constans model by itself, with the original launch condition, did not predict backwards launches were possible. Preliminary measurements showed they could in fact occur, which prompted the use of Siano for the launch condition [1][3].
- To experimentally verify the relationship, I tested 9 different values of hook angle and measured the range 5 times for each.

Summary

The floating-arm trebuchet is a more energy-efficient modern variant of the medieval trebuchet that had received little treatment in literature. In this project, I improve an already-existing Lagrangian mathematical model of the floating-arm trebuchet and combine it with a numerical analysis of projectile motion with air resistance to predict the relationship between the hook angle and the range of the projectile. To compare the theory to reality, I built my own small-scale floating-arm trebuchet and tested it with multiple values of hook angle. Similar trends were observed showing that the Lagrangian model is relatively accurate at predicting the behaviour of the floating-arm trebuchet but that it could be improved in the future by taking into account the effect of friction.

Results



- Theory and results follow the same trend.
- Peak shifted to the left and down (theory predicted an optimal hook angle of 45° for a range of 5.55 m, I observed a peak at a hook angle of 30° for an average range of 3.9±1.4 m and a maximum range of 4.8 m).
- Correcting the prediction to qualitatively account for friction in the kinematic of the arm results in a closer quantitative match with experimental measurements.

Conclusion

My theoretical model qualitatively captures the relationship between the hook angle of a floating-arm trebuchet and the range, but there is a quantitative shift which I attribute to the Lagrangian model neglecting friction during the motion of the arm.

Main areas for improvement:

- Modify the Lagrangian to account for friction in the arm's motion. Note that since friction is a non-conservative force, its addition to the Lagrangian model is not trivial [2].
- Improve quality of the build regarding alignment of parts to improve consistency.

References

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[2] V. Hirvonen, "Lagrangian Mechanics For Dummies: An Intuitive Introduction," 2023. [Online]. Available: https://profoundphysics.com/lagrangian-mechanics-for-beginners/.
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[5] L. Akiyama, "The Floating Arm Trebuchet," 2021. [Online]. Available: https://www.instructables.com/The-Floating-Arm-Trebuchet/.

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