DEVELOPING AND TESTING A LAGRANGIAN MODEL OF THE FLOATING-ARM TREBUCHET

PHYSICS

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Abstract

The floating-arm trebuchet is a modern variant of the famous medieval trebuchet that has received little attention in litterature despite being more energy-efficient than its historical counterpart. Most notably, experimental verification of mathematical models is still lacking from the literature. Here, to compare a Lagrangian mathematical model to emprical results we investigated the relationship between the hook angle of a floating-arm trebuchet and the range.

We started by developing a mathematical model that relied on Lagrangian mechanics, an alternative formulation of classical mechanics which is more appropriate than Newtonian mechanics for complex systems like the floating-arm trebuchet. Combining this Lagrangian model with a numerical analysis of projectile motion with air resistance allowed us to predict the behaviour of the floating-arm trebuchet.

To compare the theory to reality, a small scale floating-arm trebuchet was built and 9 different values of hook angle were tested 5 times each. We observed that theory and results follow the same trend but the maximum range is lower and occurs at a lower value of hook angle than predicted.

Despite the shift between theory and experiment, the similarity in the trend obtained shows that the Lagrangian model is still relatively accurate at describing the relationship between the hook angle of a floating-arm trebuchet and the range. A qualitative analysis shows that the model could be improved in the future by taking into account the effect of friction.

Introduction

The trebuchet is a medieval war machine designed to launch boulders several hundreds of meters away but nowadays, it is mainly used as a teaching tool for engineers [1] [2]. The floating-arm trebuchet is a modern variant of the famous medieval weapon of which you can find a schematic diagram in Figure 1. I find its mechanism really fascinating because the fact that the counterweight falls straight down instead of rotating around a fixed point like with the regular trebuchet allows it to convert more gravitational potential energy into kinetic energy of the projectile [3]. Despite this improvement in energy efficiency, the literature on the floating-arm trebuchet is quite sparse. In this project, I will empirically verify the theoretical model available, which to my knowledge has not been done.

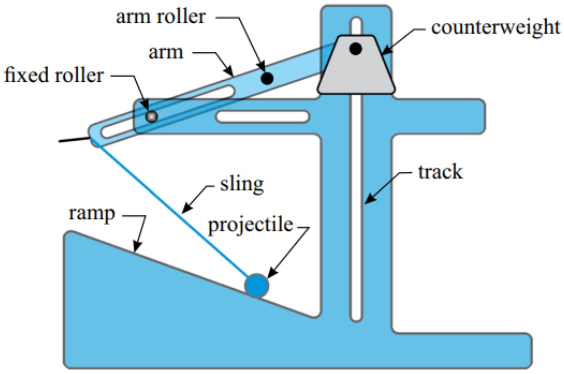


Figure 1: Schematic diagram of floating-arm trebuchet [3]. Note that the counterweight is free to move in the track, the arm is free to move along the fixed roller (which is part of the rest of the structure) but is not attached to it, the arm roller cannot go through the horizontal track but can slide along it, the sling can slide off the hook present at the end of the arm and the sling and projectile are one unit.

Research question

What is the relationship between the hook angle of a floating-arm trebuchet and the range?

Theoretical framework

The theoretical framework consists of two main parts:

- Modelling the behaviour of the floating-arm trebuchet
- Modelling the trajectory of the projectile with projectile motion including air resistance

Modelling the floating-arm trebuchet

The first part will be mainly developed using a paper by Constans that develops a mathematical model using a Lagrangian approach [3]. We can start by describing the launch sequence of the floating-arm trebuchet shown in Figure 2.

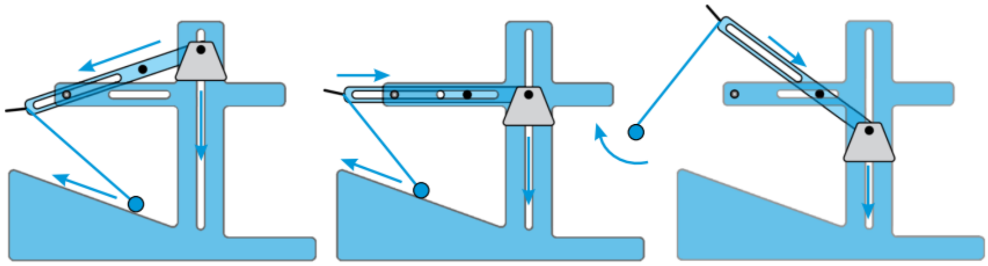


Figure 1: Launch sequence of the floating-arm trebuchet [3]

In the initial configuration, the moving arm rests on the fixed roller. As the counterweight starts to move downward, it pushes the moving arm which causes it to slide along the fixed roller, dragging the sling with it. We reach the second configuration when the arm becomes horizontal (parallel to the horizontal track). At this point, the arm roller enters in contact with horizontal track and moments later, the throwing arm only rests on the arm roller. As the counterweight continues to fall, the throwing arm resting on the moving roller rotates very quickly and creates, as Constans said, a “whipping effect” [3]. The rotation of the arm induces a rotation in the sling which ultimately lifts the projectile off the ramp. Constans then describes that the sling is released from the rest of the trebuchet when it is parallel to the hook, “in the same way as a regular trebuchet” [3]. However, Siano, who wrote a comprehensive analysis of the regular trebuchet, describes a different condition for the release of the sling which will be described in more detail later since it has no influence on the modelling of the earlier stages of the launch sequence [4].

Lagrangian mechanics is an alternative formulation of classical mechanics that is more appropriate than Newtonian mechanics for complex systems like the floating-arm trebuchet, due to its more fundamental approach [5]. Lagrangian mechanics relies on the principle of stationary action which is another way of saying that the universe wants to ‘minimise energy’. While Newtonian mechanics deals with forces to describe a system, Lagrangian mechanics uses energies (kinetic and potential), combined in a mathematical object called the Lagrangian ($\mathcal{L}$). The Lagrangian is found by taking the difference between the total kinetic energy of the system and the total potential energy of the system. The Lagrangian of the floating-arm trebuchet is shown in (1).

$$\mathcal{L} = \frac{1}{2}m_1\dot{h}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m_2\left(\dot{x_p}^2 + \dot{y_p}^2\right) - m_1gh - m_2gy_p \tag{1}$$

Before generating equations of motion however, Lagrangian mechanics requires something called 'constraint equations' which are a mathematical way of preventing the system from undergoing impossible motions. For example, the projectile cannot go through the ground. In this case, there are five constraint equations which prevent various parts of the trebuchet from phasing through one another:

- Two equations for the joint between the arm and the sling
- The arm rests on the fixed roller
- The projectile slides on the ramp
- The arm rests on the moving roller

For example, (2) represents the constraint equation vector in the first stage of the launch (in this case constraints 1,2 and 3).

$$\boldsymbol{f} = \begin{bmatrix} -L_2\cos\theta - x_p + L_3\cos\psi \\ -L_2\sin\theta - y_p + L_3\sin\psi + h \\ W\sin\theta + (H-h)\cos\theta \\ y_p \end{bmatrix} = \boldsymbol{0} \tag{2}$$

Combining the Lagrangian (1) with constraint equations (2) with the help of the method of Lagrange multipliers yields a matrix equation (3) that can be solved iteratively to calculate the behaviour of the floating-arm trebuchet until the launch.

$$\begin{bmatrix} \boldsymbol{M} & \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}}\right)^T \\ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{g} \\ \boldsymbol{\gamma} \end{bmatrix} \tag{3}$$

As was mentioned earlier, Constans wrongly interpreted the sling and hook being parallel as the launch condition [3]. This is only in the case where there is no friction between the hook and the sling and preliminary measurements showed me that friction was **not** negligible. Siano instead described that the sling will start sliding along the hook when a certain condition (shown in (4)) is reached by the angle between the arm and the hook (α) which depends on the hook angle (δ, the independent variable) and the coefficient of static friction between the hook and the sling (μ) [4].

$$\alpha = \pi + \delta - \operatorname{arccot}\mu \tag{4}$$

Modelling the trajectory of the projectile

The second part of the theoretical framework is easier to develop because projectile motion with air resistance is a very common problem. To determine which type of flow (and the type of drag) we are dealing with, we can use the Reynolds number (Re). Without going into details, it can be shown that we should use quadratic drag because our Reynolds number falls between 10^3 and 2×10^5 [6]. In the end this is described by a system of 4 differential equations shown in (5) to (8) with $k = \frac{1}{2}c_d\rho_2 A$ [7].

$$\dot{u_1} = u_2 \tag{5}$$

$$\dot{u_2} = -\frac{k}{m}\sqrt{u_2^2 + u_4^2}\,u_2 \tag{6}$$

$$\dot{u_3} = u_4 \tag{7}$$

$$\dot{u_4} = -\frac{k}{m}\sqrt{u_2^2 + u_4^2}u_4 - g \tag{8}$$

This problem famously cannot be solved analytically and therefore requires a numerical analysis. I decided to use a Python program by Hill that can conduct this analysis and allow me to calculate the range of the projectile [7].

Hypothesis

Combining the two parts of the theoretical model, we eventually arrive at Figure 3, which is then our theory of the relationship between the hook angle and the range.

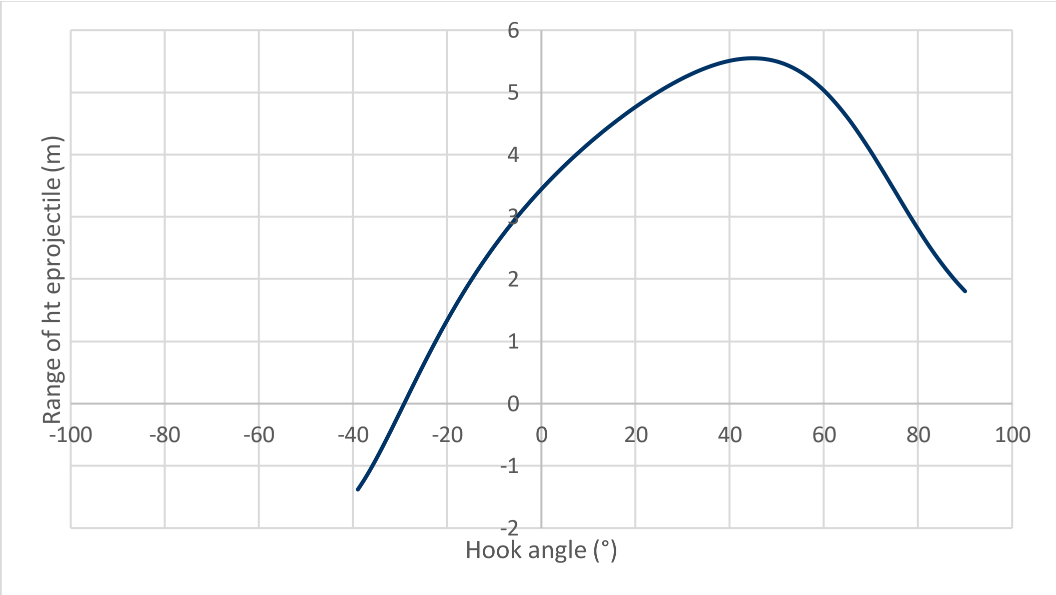
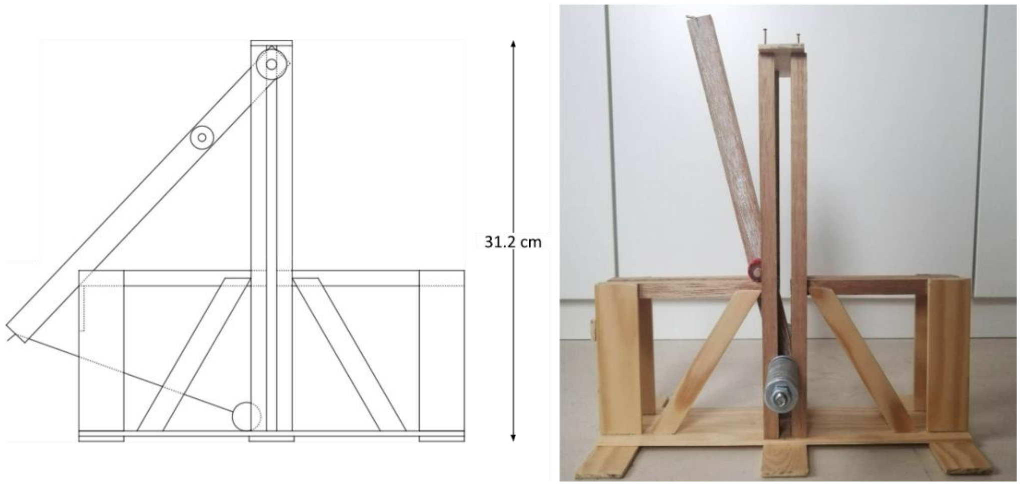


Figure 3: Graph of the range of the trebuchet against the hook angle. The hook angle ranges from -90° to 90° but values below -39° lead to a slippage of the sling at the instant the launch is initiated and therefore have no range values. The maximum range is reached at 45° with a range of approximately 5.55 m (the fact that the optimal hook angle is the same as the optimal angle for projectile motion (45°) is, to the best of my knowledge, purely coincidental).

Building the trebuchet

Now that the theoretical framework is developed, we need to build a real-life floating-arm trebuchet to test the theory. After two prototypes, it was decided that the trebuchet build was going to be based on instructions by [8]. The size was chosen to be quite small to facilitate the construction and the manipulation. Figure 4 shows the final build compared to the blueprint.



(a) Blueprint (b) Physical build

Figure 4: Blueprint of the floating-arm trebuchet compared to the final build. The blueprint is in before-launch position but the physical build is in after-launch position

Preliminary measurements

Obtaining constants for the theoretical framework

Once the trebuchet was built, a vernier calliper, a ruler and a weighing scale were used to measure the dimensions of my trebuchet which will be used as input either in the Constans simulation or the Hill program. The density of the arm (ρ_1) was calculated from other dimensions and the drag coefficient was assumed to be that of a sphere (ignoring the sling) with Reynolds number between 10^4 and 10^6 [9]. These are shown in Table 1.

Table 1: Dimensions of my trebuchet

Quantity	**Value**	**Uncertainty**
L_2 **– length of arm**	0.284 m	±0.0005 m
L_3 **– length of sling**	0.191 m	±0.0005 m
H **– vertical distance between fixed pin and origin**	0.1250 m	±0.00005 m
W **– horizontal distance between fixed pin and origin**	0.1511 m	±0.00005 m
D **– distance between arm roller and counterweight**	0.0786 m	±0.00005 m
w **– vertical thickness of arm**	0.0200 m	±0.00005 m
t **– horizontal thickness of arm**	0.0120 m	±0.00005 m
h_0 **– starting height of counterweight**	0.275 m	±0.0005 m
β **– angle of ramp**	0 rad	NA
m_1 **– mass of counterweight**	0.370 kg	±0.0005 kg
m_2 **– mass of projectile**	0.014 kg	±0.0005 kg
m_3 **– mass of arm**	0.081 kg	±0.0005 kg
ρ_1 **– density of arm** $\left(\frac{m_3}{L_2wt}\right)$	1188 kg/m^3	±17 kg/m^3
d **– diameter of projectile**	0.0221 m	±0.00005 m
c_d **– drag coefficient of projectile**	0.47	NA

The coefficient of friction between the hook and the sling (μ, used in (4)) was calculated to be 0.31 ± 0.04.

For the Hill program we also need a value for air density which was calculated from weather data to be approximately 1.29 kg/m^3.

Experimental method

Setup

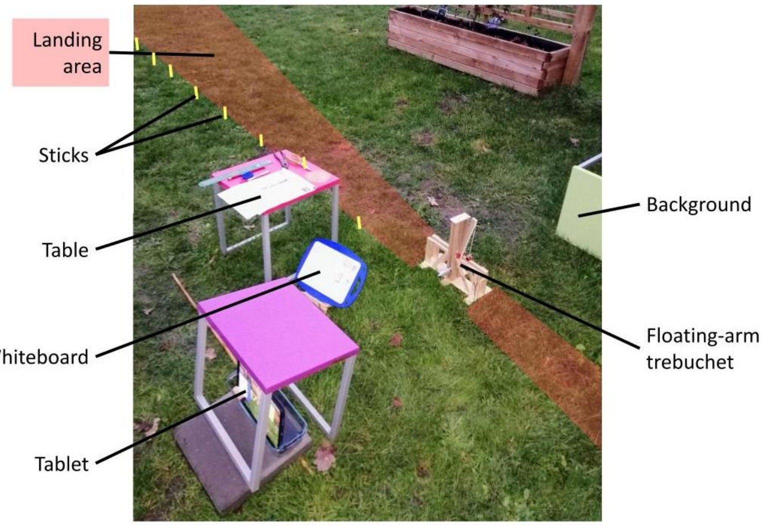


Figure 5: Picture of the setup. The sticks were hard to see on the picture so yellow lines have been added to clearly show where they are.

Procedure

1. Set up as shown in Figure 5. The ground needs to be flat and the trebuchet level. The sticks are placed every 50 cm from the centre of the trebuchet in both directions using the measuring tape (5 m in forward direction and 2.5 m backward).
2. Using the pliers and the protractor, bend the hook so that is has a certain angle.
3. Write the ongoing trial on the whiteboard and start the video on the tablet.
4. Lift the arm until it touches the top and hold it.
5. Install the sling on the hook and pull back the projectile on the ramp.
6. Hold the arm up against the top with one finger only, and remove it in a swift movement. The projectile should launch (if it is a misfire, repeat this trial).
7. Stop the video, measure and record the range using the sticks as reference points and the ruler for a range value down to 0.1 m.
8. Repeat steps 3-7 five times for each hook angle value.
9. Repeat steps 2-8 for a different hook angle value.

Results and analysis

The results of the experiment are shown in Table 2.

Table 2: Results of the experiment

Hook angle δ (±5°)	**Range R (m)**						**Uncertainty (m)**
	Trial 1	**Trial 2**	**Trial 3**	**Trial 4**	**Trial 5**	**Average**	
-45	-1.2	-0.7	-2.2	-2.1	-1.4	-1.5	±0.8
-30	-0.8	-0.6	-0.7	-0.4	-0.4	-0.6	±0.2
-15	1.6	-0.2	1.0	-0.8	2.4	0.8	±1.6

0	2.6	3.5	2.5	1.5	1.4	2.3	±1.1
15	3.0	3.3	3.0	4.0	4.0	3.5	±0.5
30	3.4	2.0	4.8	4.7	4.5	3.9	±1.4
45	4.4	2.0	3.9	4.2	3.7	3.6	±1.2
60	1.8	2.4	1.9	2.1	3.2	2.3	±0.7
75	2.9	2.0	1.7	1.0	1.6	1.8	±1.0

We can notice that the range was never very consistent (except δ=-30°), which led to large uncertainties using the extreme value method. This is probably caused by imperfections in the construction.

Then, looking at the numbers, we see that δ=30° led to both the highest average range (3.9 m) and the trial with the longest range (4.8 m). However, as can be seen in Figure 3, theory predicted the maximum to occur at 45° and have a value of 5.55 m. If we plot the results against theory, we can visualise these discrepancies more easily, as is shown in Figure 6.

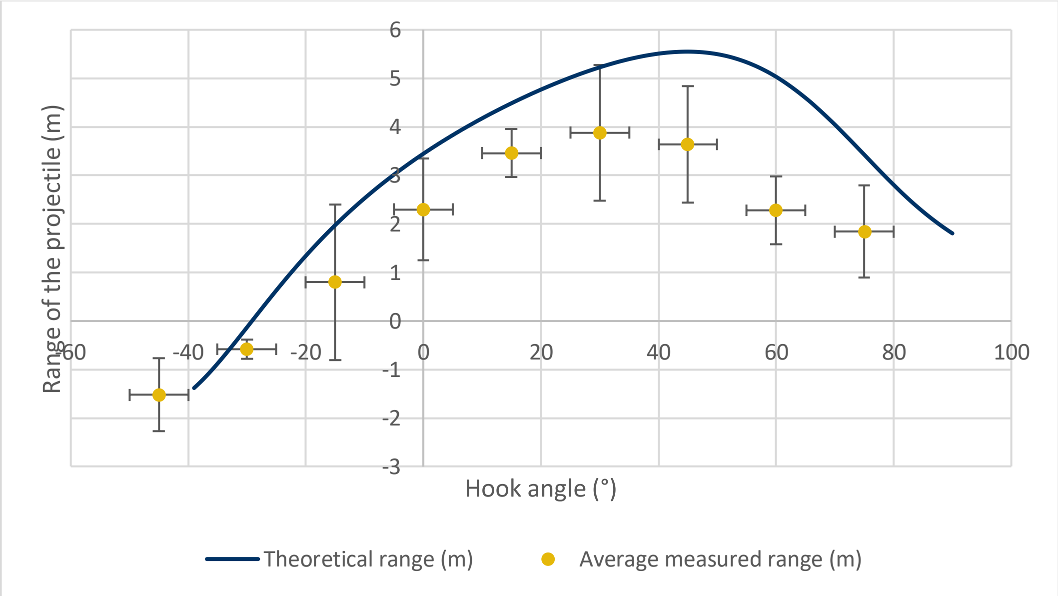


Figure 6: Plot of the experimental results with error bars alongside theory

As you can see in Figure 6, despite the large error bars, it seems the experimental results follow the same kind of trend as the theory. However, we notice the peak is lower and slightly shifted to the left. Similarly, the real trebuchet had successful launches for δ=-45° while theory predicted the lowest hook angle possible would be -39°.

Without going into details, due to the effect of friction (unnaccounted for in my model), it can be reasoned out that if we want to modify theory to get a better prediction of the real-world relationship between hook angle and range, we should shift the theoretical range to the left and decrease it by a multiple of the time spent in the trebuchet squared. By manually playing with the coefficients, Figure 7 could be obtained, which shows the adapted theory. It is visually a closer match with reality and passes through all the error bars.

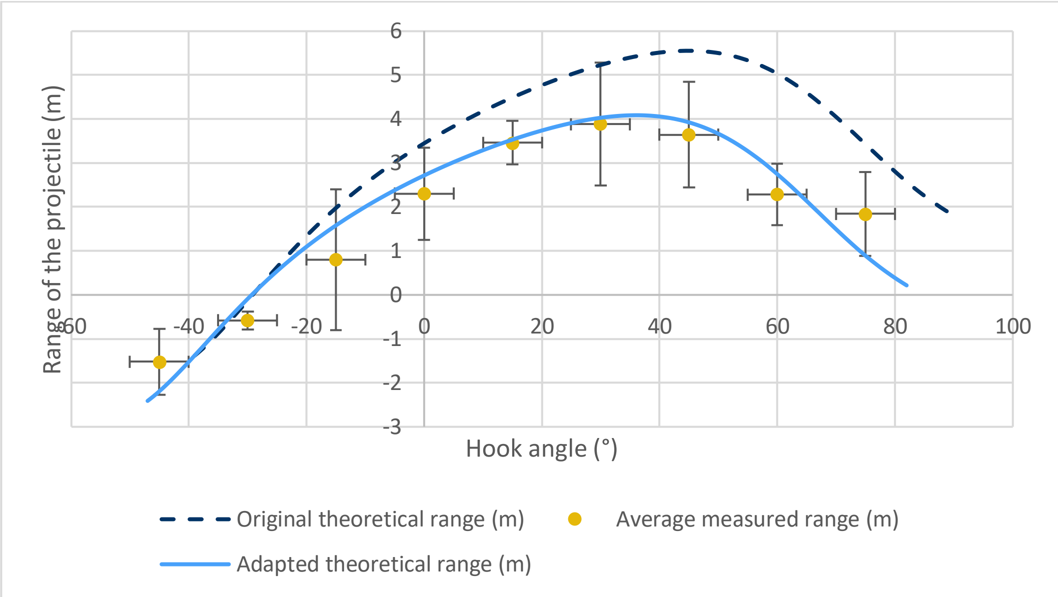


Figure 7: Plot of the experimental results alongside original theory (shown in **Error! Reference source not found.**) and adapted theory. Notice how the adapted theory is visually a closer match to the measurements than the original theory, and passes through all the error bars.

Conclusion

The research question was "What is the relationship between the hook angle of a floating-arm trebuchet and the range?". Using a Lagrangian mathematical model and a numerical analysis of projectile motion, we were able to predict this relationship with Figure 3**.** Although the theory predicted an optimal hook angle of 45° for a range of 5.55 m, we observed a peak at a hook angle of 30° for an average range of 3.9±1.4 m and a maximum range of 4.8 m. Yet, the similarity in the kind of trend obtained both in the theory and the results as demonstrated by Figure 6 shows our theoretical framework is still relatively accurate at describing the relationship between the hook angle of a floating-arm trebuchet and the range. Finally, Figure 7 shows that the discrepancy between theory and results could be explained by friction which was not taken into account in the mathematical model.

Discussion

In the analysis of the results, we obtained Figure 7 where we artificially modify the original theory curve to try to explain the shift in the results due to friction. However, even if this process leads to visually appealing results, it is completely qualitative (what "looks" good). Obviously, a more accurate way of including friction quantitatively would be to account for it in the Lagrangian or in the constraints from the beginning, but this requires further research. Also, since friction is a non-conservative force, it cannot be trivially added to the Lagrangian model like constraint forces and a different approach might be preferable [5].

Then, as is visible in Table 2, the error bars in the results are large which means the range is not very consistent. This indicates that the quality of the build was not very high because even starting in seemingly the same conditions, the trebuchet will still react differently and have a different range. To decrease uncertainties, the physical trebuchet should be made with more care regarding the alignment of the parts for example as this would improve the consistency. A future researcher might also want to consider using another building material than wood which tends to be slightly flexible.

In case the results were very different from the theory, videos of the launches were made to potentially identify where the theoretical framework would have gone wrong. Fortunately, no large discrepancy was found so the videos were not used. Still, maybe an in-depth analysis of the videos could help quantify the friction mentioned above and help design a new mathematical model.

References

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