WATER SPIRAL

A STUDY ON CIRCULAR, ELLIPTICAL AND SWIRLING JET FLOWS

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Abstract

The present work investigates circular, elliptical and swirling jet flows in the laminar Rayleigh regime. A basic equation for the free surface of inviscid circular jets is derived from Bernoulli's principle. For the elliptical jet, a set of nonlinear partial integro-differential equations (the Cosserat equations) is derived directly from the Navier-Stokes equations, and a novel derivation is performed for the present work. Based on the Cosserat equations, asymptotic scaling yields an equation for the free surface of elliptical jets. Furthermore, the equations are linearized with perturbation theory and a dispersion relation is derived that yields the wavelength of elliptical jets. The phenomenon of swirling jet flows with elliptical cross-sections (water spirals) is investigated both experimentally and theoretically as the main novelty of this work. Sophisticated passive nozzles have been designed and 3D-printed to reproduce the phenomenon. Experimental data is obtained for the circular, the elliptical and swirling jets and compared with theory. The results show very good agreement in general. A method to clearly distinguish the different cases is developed based on the jet wavelength.

Introduction

The basis of this research stems from the ninth problem of the International Young Physicists' Tournament (IYPT) 2022. The problem number nine is called "Water Spiral" and the task is stated as follows:

> If a stream of liquid is launched through a small hole, then under certain conditions it twists into a spiral. Explain this phenomenon and investigate the conditions under which the spiral will twist.
>
> (IYPT, 2022)

The investigation of circular and elliptical jets without a swirling base flow are essential foundations of the swirling jet and the reader ought to consider this research as an extension to the task statement issued by the IYPT.

Imposing swirl on the base flow of an elliptical liquid jet changes its physical characteristics remarkably. This phenomenon raises various research questions, in particular: (I) What are the physical effects that govern the behavior of swirling jets? (II) What are the underlying conditions that cause a jet to twist into a spiral? (III) How do the jet characteristics behave under parameter variation? These swirling jets have not been discussed in literature, so the present work also involved a major, experimental challenge: (IV) How can the swirling jets be reproduced to conduct accurate experimentation? To answer those questions, a thorough investigation of the more basic cases, i.e., the circular and elliptical jets, was required as well.

Theory

When opening the tap, a water jet is ejected through the circular nozzle into the sink. Certainly, everyone has noticed that the shape of the jet narrows down the stream. The challenge when analyzing circular jets is to find the shape of the jet which is called the free surface. When the orifice is elliptical, the free surface of the jet exhibits an oscillatory shape where the semiaxes of the jet interchange. This oscillation is called axis-switching and the jets are commonly referred to as elliptical jets. The main characteristics of elliptical jets are the aspect ratio of the elliptical cross-section, the free surface, and the wavelength of the oscillations. By imposing a swirl on the elliptical jet, we finally obtain the sought-after water spiral which we refer to as the swirling jet. The method that was used in this work comprised a nozzle with a twisted channel. The swirling jet, emitted from an elliptical orifice, is not discussed in literature.

Figure 1: Diffferent jets: (a) circular jet, (b) elliptical jet, (c) swirling jet

Qualitative Analysis

To understand the phenomenon of swirling water jets, it is essential to first provide the foundation of the elliptical jets without a swirling base flow. The relevant properties that characterize elliptical jets are the shape of the free surface and the wavelength.

When a fluid issues from an elliptical orifice, the cross-section of the jet starts to oscillate down the stream of the jet. The elliptical cross-section deforms towards a circular shape and then overshoots into an ellipse again but with interchanged semi-axes. This results in an oscillation and the phenomenon is commonly referred to as axis-switching. Essentially, axis-switching is based on a competition between surface tension and inertia. Surface tension tries to reduce the surface-to-volume ratio of a fluid body. Therefore, cohesion forces tend to change the cross-section into a circle since a liquid column with an elliptical cross-section has a larger surface-to-volume ratio than a liquid column with a circular cross-section. As the semi-major axes accelerate towards the center, the semi-minor axes must evade laterally because of mass continuity. Then again, surface tension decelerates the inertia and eventually, we end up with the same ellipse, but with switched axes. The forces acting in the cross-sections are qualitatively demonstrated in Fig. 2. Down the jet, the axis-switching either gets damped out because of viscous dissipation or ends with the breakup of the jet into droplets.

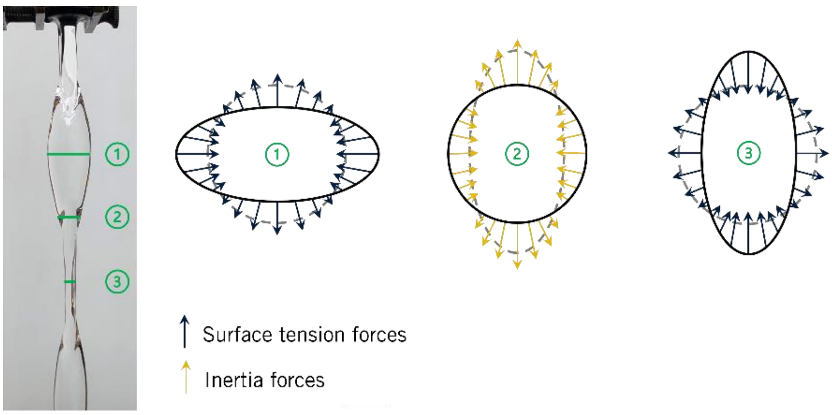


Figure 2: Forces acting on the jet cross-section

In literature, no documentation of swirling jets with elliptical cross-sections can be found. The following qualitative theory is based on observations of experiments.

When imposing a swirling base flow on the elliptical jet, one usually obtains a water spiral. The 3D-printed nozzle forces a swirl on the jet. In the cross-section of the elliptical water jet without swirl, the forces that dominate are the capillary and lateral inertia forces. Of course, the same forces are apparent in swirling jets with elliptical cross-sections. In the case of a swirling jet, we additionally have forces due to the rotation of the cross-section. Considering a rotating frame of reference, the fluid elements in the cross-section experience a force that pulls them outwards. Surface tension drives the elliptical cross-section towards a circular shape. Angular momentum $L = I\omega$ is conserved and since the moment of inertia decreases when [1]the jet becomes more circular, the angular frequency and therefore centrifugal force increases. We now have a competition between surface tension and the counteracting centrifugal effects. This competition prohibits axis-switching from occurring, but we still have a slight oscillation due to the centrifugal and surface tension effects.

Quantitative model

In the following, $\phi_{1,2}$ are the semi-axes of the jet, z is the axial distance along the jet axis, $\tilde{\lambda}$ is the dimensionless jet wavelength and We is the Weber Number which represents the relative effect of surface tension to the inertia. The coordinate system and axes are qualiatively depicted in Fig. 3.

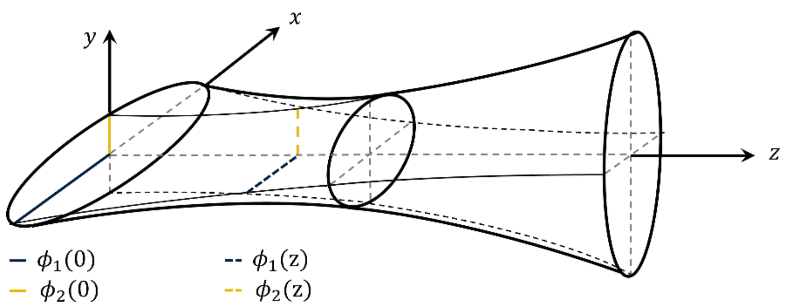


Figure 3: Qualitative depiction of the elliptical jet wih coordinate system and semi-axes marked

To find an appropriate mathematical description of our problem, one can construct a model based on the Navier-Stokes equations. Previously, similar equations have been derived from different continuum conervation equations [1].

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla\cdot\boldsymbol{\sigma}_{\mathrm{ij}} + \rho\mathbf{g}, \qquad \boldsymbol{\sigma}_{\mathrm{ij}} = p\boldsymbol{\delta}_{\mathrm{ij}} + 2\mu\boldsymbol{\epsilon}_{\mathrm{ij}} \tag{1}$$

The Navier-Stokes equations are Newton's second law of motion ($F = ma$) for fluids. The term on the left-hand side of the left equation represents mass times acceleration and the terms on the right-hand side are the sum of forces. The equations are integrated over the jet cross-section and with the free surface boundary conditions, we end up with the Cosserat equations in the x, y and z directions [2]:

$$\begin{aligned}
&\frac{1}{4}\pi\rho\phi_1^3\phi_2\left(\frac{\partial\zeta_1}{\partial t}+v_z\frac{\partial\zeta_1}{\partial z}+\zeta_1^2\right)+2\mu\pi\phi_1\phi_2\zeta_1 = p+\phi_1\phi_2 h(\phi_1,\phi_2)+\frac{1}{4}\mu\pi\frac{\partial\left(\phi_1^3\phi_2\frac{\partial\zeta_1}{\partial z}\right)}{\partial z}\\
&\frac{1}{4}\pi\rho\phi_2^3\phi_1\left(\frac{\partial\zeta_2}{\partial t}+v_z\frac{\partial\zeta_2}{\partial z}+\zeta_2^2\right)+2\mu\pi\phi_2\phi_1\zeta_2 = p+\phi_2\phi_1 h(\phi_1,\phi_2)+\frac{1}{4}\mu\pi\frac{\partial\left(\phi_2^3\phi_1\frac{\partial\zeta_2}{\partial z}\right)}{\partial z}\\
&\pi\rho\phi_1\phi_2\left(\frac{\partial v_z}{\partial t}+v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z}-\phi_2\frac{\partial\phi_1}{\partial z}h(\phi_1,\phi_2)-\phi_1\frac{\partial\phi_2}{\partial z}h(\phi_1,\phi_2)+2\mu\pi\frac{\partial\left(\phi_1\phi_2\frac{\partial v_z}{\partial z}\right)}{\partial z}
\end{aligned} \tag{2}$$

with

$$\begin{aligned}
h(\phi_1,\phi_2) = \sigma\int_0^{2\pi} \Bigg[&(\phi_1^2\sin^2\theta+\phi_2^2\cos^2\theta)\Bigg(\frac{\partial^2\phi_1}{\partial z^2}\phi_2\cos^2\theta\\
&+\frac{\partial^2\phi_2}{\partial z^2}\phi_1\sin^2\theta\Bigg) - \phi_1\phi_2\Bigg(\left(\frac{\partial\phi_1}{\partial z}\right)^2\cos^2\theta\\
&+\left(\frac{\partial\phi_2}{\partial z}\right)^2\sin^2\theta+1\Bigg)\Bigg] - \phi_1\phi_2\Bigg(\left(\frac{\partial\phi_1}{\partial z}\right)^2\cos^2\theta\\
&+\left(\frac{\partial\phi_2}{\partial z}\right)^2\sin^2\theta+1\Bigg)\Bigg]\Bigg[\left(\phi_1\frac{\partial\phi_2}{\partial z}\sin^2\theta+\phi_2\frac{\partial\phi_1}{\partial z}\cos^2\theta\right)\\
&+\phi_1^2\sin^2\theta+\phi_2\cos^2\theta\Bigg]^{-1.5}\cos^2\theta\,\mathrm{d}\theta
\end{aligned} \tag{3}$$

We can linearize those equations with Taylor expansions and perturbation theory to get an equation for the free surface of the elliptical jet [3]:

$$\left[1+\phi_1^4\left(\frac{2\,z}{\mathrm{Fr}}+1\right)\right]\frac{\partial^2\phi_1}{\partial z^2}-\frac{2}{\phi_1}\left(\frac{\partial\phi_1}{\partial z}\right)^2+\frac{4\phi_1^3}{\mathrm{We}}\left[h^*(\phi_2,\phi_1)-h^*(\phi_1,\phi_2)\right]+\frac{1}{\mathrm{Fr}}\left[\phi_1^4-\left(\frac{2\,z}{\mathrm{Fr}}+1\right)^{-1}\right]$$
$$-\frac{2\phi_1}{\mathrm{Fr}^2}\left(\frac{2\,z}{\mathrm{Fr}}+1\right)^{-2}=0 \tag{4}$$

and the wavelength of the elliptical jet [4]:

$$\tilde{\lambda}=\frac{2\pi}{\sqrt{6}}\sqrt{\mathrm{We}-2} \tag{5}$$

A detailed derivation is undocumented and tedious but is performed for the present work. For the wavelengths of swirling jets, analogous proportionalities to Eq. 5 based on the Weber Number are derived.

Experimental Apparatus

To get a continuous liquid flow, a circuit with a membrane pump and a pulsation dampener was developed that guarantees a continuous flow rate. A photograph of the circuit can be seen in Fig. 4.a and a diagram in Fig. 4.b.

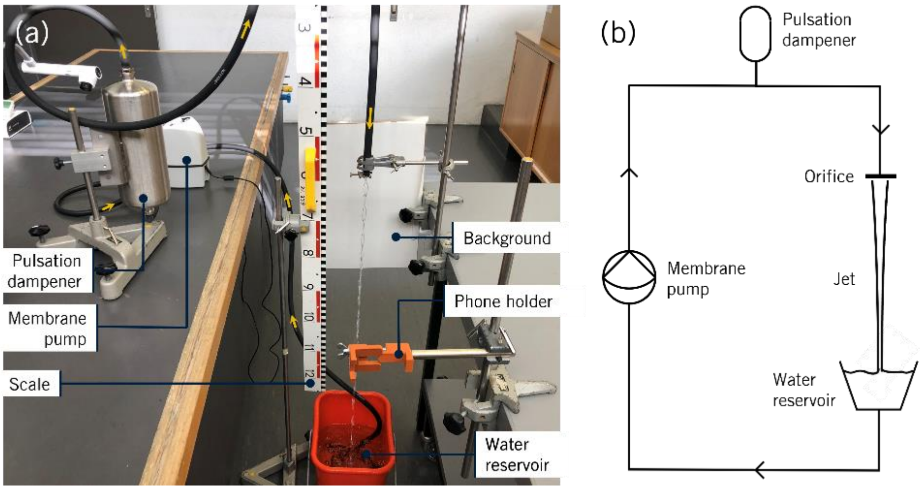


Figure 4: (a) Image of the flow circuit with labelled components, (b) diagram of the flow circuit

The main experimental challenge of this project was designing a system that creates swirling jets. In literature, swirling jets with elliptical orifices generated with a passive system (not a motor) have never been documented. The main challenge when developing the current experimental setup was to keep the jet laminar. The solution was a 3D-printed nozzle system that allows variable swirl as (Fig. 5.b). The chamber eliminates initial disturbances by making the flow turbulent. The jet enters a straight channel to develop a laminar velocity profile again and at the end, we impose a swirl with the twisting channel (Fig. 5.a).

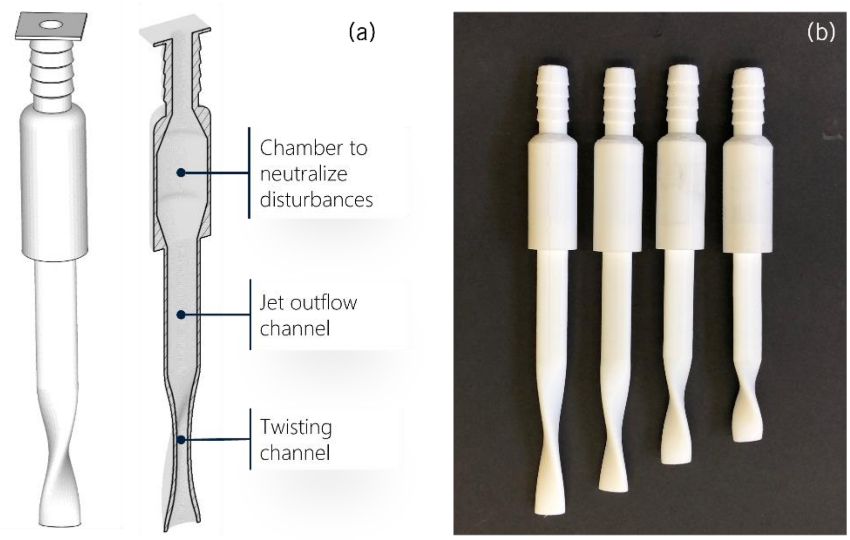


Figure 5: (a) CAD sketch and cross-section of a nozzle with labelled sections, (b) several 3D-printed nozzles with various twisting channel lengths

Results and Discussion

The theory for the free surface of elliptical jets (Eq. 4) is now compared to experiments and shows good agreement (Fig. 6). It allows a thorough analysis of the non-linearities in the system, and we can find when axis-switching occurs and when it is simply dampened out by surface tension. This is crucial since axis-switching is an essential condition for a swirling jet to occur.

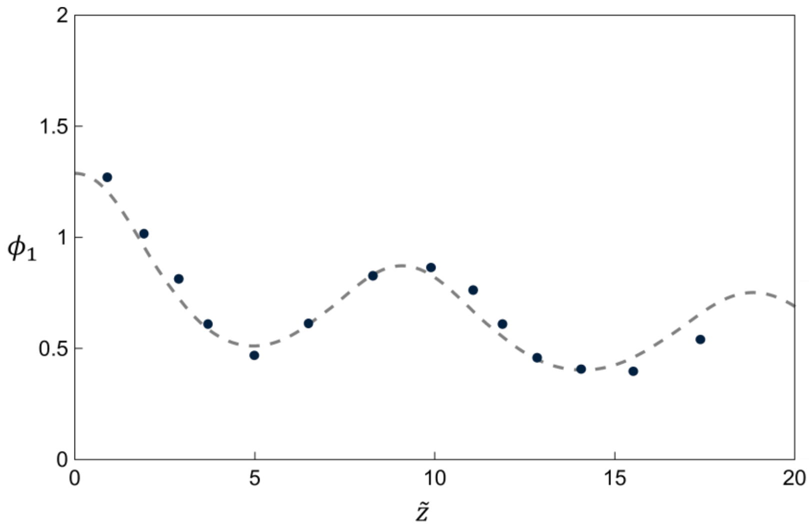
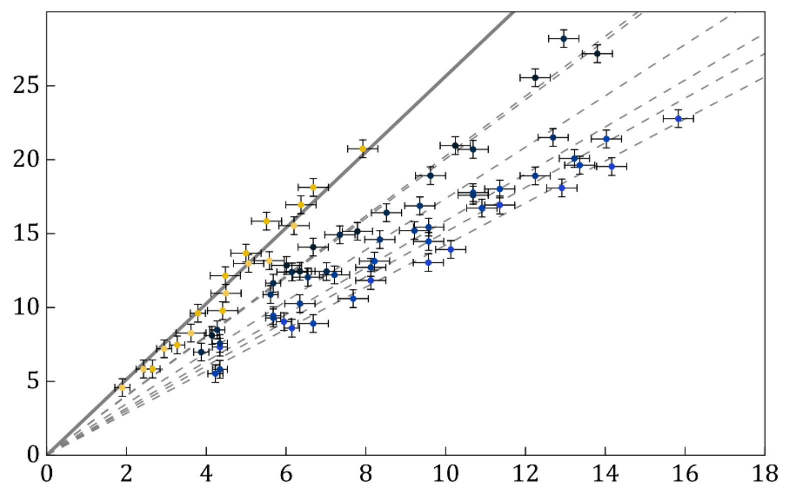


Figure 6: Free surface of an elliptical jet. The grey-dashed line represents the numerical, implicit solution of Eq. 4 and the blue dots are datapoints from image tracking of the free surface. The error bars due to image tracking are not visible because they are smaller than the dots. The units of the axes are dimensionless.

Furthermore, the wavelengths of elliptical and swirling can be measured for different nozzles, and we generally see a decrease in wavelengths with increasing swirl Fig. 7. The details are thoroughly investigated in the present work. This allows us to assess whether the axis-switching behavior or the spiral is dominant. This distinction method for the elliptical and swirling jets based on the wavelengths is another key finding since for small amplitudes, it is impossible to see by eye which case occurs.



Figur 7: Plot of $\tilde{\lambda}$ against We. Yellow datapoints: two datasets of elliptical jets with different aspect ratios (1.66, 3.13). Blue datapoints: six datasets of swirling jets with different initial swirls. Thick, gray line: linearization of Eq. 5. Dashed, gray line: proportionality fit to each swirl dataset.

Closure

Summary and Conclusion

First, different jet regimes were qualitatively introduced, and it was stated that the work is limited to the Rayleigh regime. Afterwards, an equation for inviscid circular jets based on Bernoulli's theorem was derived. The equation was numerically solved and showed high accuracy when compared with experimental data of water jets with a low flow rate. The trivial solution only considering gravitational acceleration was not accurate which makes it clear that the axial surface tension effects are important for low flow rates. Under appropriate assumptions, a reasonable base flow ansatz and the use of the free surface boundary conditions, Cosserat equations were derived for the elliptical viscous jets. Derivations cannot be found anywhere in literature and have therefore been calculated in this paper. Asymptotic scaling was performed which is the negligence of axial surface tension forces. The reduced equation was numerically solved and showed good agreement for high flow rates in the Rayleigh regime. Perturbation methods were performed to linearize the Cosserat equations for a circular jet with varicose disturbances. This is linked to the elliptic jet and an equation for the wavelength of the axis-switching is derived. Experiments show that the linearization is accurate even though disturbances are big compared to the initial radius. The trivial solution based on a simple proportionality is equally accurate in our measurement range. The phenomenon of the water spiral is interpreted as an elliptical jet with a swirling base flow and is reproduced with 3D-printed nozzles. A CFD simulation was run on a nozzle to examine the internal flow development. The nozzles were developed and then tweaked to extend our measurement regime for which the jet must remain laminar. The nozzle design was thoroughly examined and with a process of trial and error optimized. According to our basic theory, a linear fit is applied to the data that is indeed accurate. Although under low flow rates, low initial swirl, and a small aspect ratio, telling whether axis-switching or the spiral dominates is impossible. Nevertheless, it was found that the wavelength of the jets can serve as a method to distinguish the jets.

Outlook

Liquid jets are the subject of several industrial applications. A study on jets is of basic interest for, *e.g.,* inkjet printing, the coating of optical fibers and polymer mixing. It has been proven that elliptical jets outperform jets with circular orifices when it comes to mixing and spraying due to the shorter breakup length and better mixing characteristics. It seems intuitive that swirling jets could even outperform elliptical jets when it comes to applications such as mixing or spraying. We impose the swirl on the jet with a passive device, the nozzle, instead of a motor what has commonly been done to achieve rotating jets. A motor is an external power source, but the nozzle is not. Therefore, implementation into certain devices and manufacturing processes might be applicable to increase efficiency.

References

[1] D. A. Caulk and P. M. Naghdi, "On the Onset of Breakup in Inviscid and Viscous Jets," *Journal of Applied Mechanics,* vol. 46, no. 2, pp. 291-297, 1979.

[2] S. Gu, L. Wang and D. L. S. Hung, "Instability evolution of the viscous elliptic liquid jet in the Rayleigh regime," *Physical Review E,* vol. 95, no. 6, 2017.

[3] S. E. Bechtel, "The Oscillation of Slender Elliptical Inviscid and Newtonian Jets: Effects of Surface Tension, Inertia, Viscosity, and Gravity," *Journal of Applied Mechanics,* vol. 56, no. 4, pp. 968-974, 1989.

[4] G. Amini and A. Dolatabadi, "Axis-switching and breakup of low-speed elliptic liquid jets," *International Journal of Multiphase Flow,* vol. 42, pp. 96-103, 2012.