Cubic Pell's Equation

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Abstract

This is an initial study of Pell's equations of higher degree, which is an open problem in Number Theory. The first step is to investigate the Pell's equation of the form $x^3 - dy^3 = 1$. Later, we consider the form $N(\theta) = x^3 + cy^3 + c^2z^3 - 3cxyz = 1$, where $\theta^3 = c$ for some non-perfect cube integer c. For this form, it is found that for some certain c values, solutions can be generated by an algorithm similar to that for the quadratic Pell case. However, this algorithm does not work for all c values, for example $c = 15$ and $c = 16$. Investigating these equations involves literature studies and computational researches in Wolfram Mathematica as well as MatLab R2011a. The cubic Pell's equation has possible applications within Approximation Theory and Cryptography.

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Contents

List of Symbols

Symbols	Meaning and Example
$\mathbb{Z}$	The set of all integers. $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$.
$\mathbb{Q}$	The set of all rational numbers $\frac{p}{q}$, p is any integer and q is a non-zero integer.
$a \in A$	a is an element in A, e.g. $3 \in \mathbb{Z}$.
$b \notin A$	b is not an element in A, e.g. $\frac{3}{2} \notin \mathbb{Z}$.
$x_1 \vee x_2$	Set logical *or* For example, $x^2 - 3x + 1 = 0$ has the roots x_1 or x_2. We write $x = x_1 \vee x_2$. This is true if x_1 is a solution, if x_2 is a solution and if both x_1 and x_2 are solutions.
$A \Longrightarrow B$	A implies B. For example $x > 2 \Longrightarrow x^2 > 4$. Note that $x^2 > 4$ does not imply $x > 2$.
$A \Longleftrightarrow B$	A is equivalent to B. For example $x - a = 0 \Longrightarrow x = a$ and $x = a \Longrightarrow x - a = 0$, then $x - a = 0 \Longleftrightarrow x = a$.
$a \leqslant b$	$a = b \vee a < b$.
$\lvert a \rvert$	The absolute value of a, e.g. $\lvert -3 \rvert = 3$.
∞	Infinity.
$a \to \infty$	a approaches infinity.
$A \longrightarrow B$	The input of an object A gives an output as an/the object B. For example if $x = 2$ in $x \longrightarrow x^2$, we will receive the output 4.
$\pm x$	Either x or $-x$.
$a \mid b$	a divides b, i.e. $a = bc$ for integer c if a and b are integers.
$a \nmid b$	a does not divide b, i.e. $a = bc + d$ for $0 < d < b$, a, b, c, d are integers.
$a \equiv b \pmod{c}$	$a = cd + b$ for integers a, b, c and d, $b < c$. $3 \equiv 1 \pmod{2} \Longleftrightarrow 3 = 2 \cdot 1 + 1$.
$a \in [b, c]$	$b \leqslant a \leqslant c$
$\blacksquare$	In this report, this means Q.E.D. or End of Proof.

1 Introduction

I have been inspired to work with this problem by Kjell Elfström, who introduced me to the quadratic Pell's equation and continued fractions. At first, I was more interested in continued fractions but I soon found a growing interest in the actual equation because I did not know much about it. Pell's equation of higher degree in general can be describe as an interesting meeting point of our ancient and modern Mathematics.

The quadratic Pell's equation is a Diophatine equation of the form

$$x^2 - dy^2 = 1, \tag{1}$$

where d is a non-square integer, i.e. $d \neq n^2$ for any integer n. Solution pairs (x, y) for which $x, y \in \mathbb{Z}$ are sought. The focus of this report is to investigate two possible cubic versions of this equation, namely:

1. $x^3 - dy^3 = 1$, for some non-cubic integer d. (2)

   a. What can be said about this equation? More specifically, what are the properties of this equation – if there are any?

   b. A strategy/strategies to solve this equation and its weakness/their weaknesses.

   c. Why should/should not the given equation be considered as the cubic Pell's equation?

2. $x^3 + cy^3 + c^2y^3 - 3cxyz = 1$, for some non-cubic integer c. (3)

   a. The definition and motivation to this version of cubic Pell's equation.

   b. Properties of this equation.

   c. A strategy/strategies to solve this equation.

Here, equation (2) has the same form as the original Pell's equation, while equation (3) is defined analogously as the quadratic case. It will be obvious that equation (3) is the proper cubic Pell's equation because it has more similarities with the quadratic Pell. Since this is a very challenging problem, finding an efficient method for solving it is not to be expected within the time limit of the Project Course spring-term 2012. Nevertheless, there are reasons, based on the results of this project, to believe that this equation will be solved and applied in the future.

One major difficulty encountered during the project process is the lack of understandable or accessible materials treating higher degrees of Pell's equation. Subsequently, approximately 80% of this project has been based on E. Barbeau's *Pell's Equation*, 2003, chapter 5 and 7. Other works that have enriched my understandings are J.Pang's bachelor's thesis, Algebra course book and IMO's training materials. The algorithm described in section 4.3.3. has been encoded by Lyar F. Nguyen at HUST. Proposals for future work and actions to develop this project are specified in section 6. For the sake of reproducibility, a list of syntax and codes that I used in Mathematica Wolfram 7 and MatLab R2011a are included in the Appendix.

2 Background

Since the equation (3) is based on the definition of the quadratic Pell's equation, some knowledge about the equation $x^2 - dy^2 = 1$ is an important piece of jigsaw to one's holistic view on this problem. Also, Pell's equation brilliantly exemplifies the rich and exciting history of Mathematics.

2.1 Historical Notes about the Quadratic Pell's Equation

If Number Theory is the Queen of Mathematics [1], then Pell's equation must be the most important piece of jewelry to this Queen.

The equation $x^2 - dy^2 = 1$ is believed to appear very early in the history of Mathematics. The Greeks were fascinated in $x^2 - 2y^2 = 1$ because of its strong connection to the approximation of $\sqrt{2}$ [2]. Both the Father of Geometry, Euclid, and the Father of Algebra, Diophantus were involved in the early studies of Pell's equation. Archimedes too was suspected to have had an interest in Pell's equation but this is a weak link. What he contributed is the so called *Archimedes' Cattle Problem*, which has challenged generations of mathematicians and Computational Science.

For some reason, Pell's equation fell into oblivion for a long period of time. The equation was rediscovered and was extensively studied by Pierre de Fermat (1601 – 1665). However, the first European mathematician who solved the Pell's equation was William Brouncker (c.1620 – 1684), who also actively developed the theory of continued fractions. Having read this far, one might wonder what connection the name Pell has to this equation. The answer is: none at all. When the giant Euler (1610 – 1665) made his way into the playground of Mathematics, he mistook William Brouncker for John Pell (1611 – 1685), an English mathematician. Besides this, Euler challenged Europe to find a solution to the particularly difficult equation $x^2 - 61y^2 = 1$. However, this is not "supreme wisdom" in comparison to the earlier mentioned *Cattle Problem*, discovered by Lessling (1729 - 1781). If Euler had been born one generation later, he might have abandoned his own challenge since one solution for Archimedes' problem takes 47 pages of computer printout, 206 545 digits in total. Notable historical attempts to achieve "supreme wisdom" have been made by various mathematicians and among them was A. Amthor, who finally won the prize of "supreme wisdom" in 1880 not only to him but also to the field of Computational Science.[3].

While the 18th-century European mathematicians occupied themselves with Pell's equation, they never knew that an Indian astronomist and mathematician, Brahmagupta (598 – 668 CE), had developed a different and sufficient method. Yet, we will not focus much on Brahmagupta's method in section 2.2. Notably enough, not many, even Brahmagupta, Euler and Fermat themselves, ever bothered to prove the existence of a fundamental solution for this equation. They merely developed an algorithm that only worked if a fundamental solution existed. This is where Lagrange (1736 – 1813) came into the picture. In 1768, he proved that there always exists a fundamental solution for Pell's equation. Another proof was introduced by Dirichlet (1805 – 1859), using the pigeonhole principle.

Pell's equation is now applied in Approximation Theory and Cryptography [6].

2.2 A Short Overview of the Quadratic Pell's equation

The original Pell's equation has the form $x^2 - dy^2 = 1$, where d is not a perfect square. This is a Diophantine equation, i.e. integer solution pairs (x, y) are sought. One trivial solution is $(\pm 1, 0)$. Factorization of the left side gives

$$x^2 - dy^2 = (x - y\sqrt{d})(x + y\sqrt{d}). \tag{4}$$

With modern mathematical terminology, this is described as a factorization in the quadratic field $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} : \ a, b \in \mathbb{Q}\}$. A number $x + y\sqrt{d}$ has the surd conjugate $x - y\sqrt{d}$. The product of a number in this set and its surd conjugate gives a norm. Consequently, the quadratic Pell's equation can be treated as a norm equation in this field, $N(x + y\sqrt{d}) = 1$. Since x and y are integers, the subset $\mathbb{Z}(\sqrt{d}) = \{a + b\sqrt{d} : \ a, b \in \mathbb{Z}\}$ is more relevant. If $N(x + y\sqrt{d}) = 1$, then $x + y\sqrt{d}$ is a unit. A unit is the number α that has its reciprocal, i.e. $\frac{1}{\alpha}$, also in $\mathbb{Z}(\sqrt{d})$.

The norm is multiplicative, i.e. if α and β are numbers in $\mathbb{Z}(\sqrt{d})$, then $N(\alpha\beta) = N(\alpha)N(\beta)$. This is an important property because it can be used to prove that $N(\alpha^n) = N(\alpha)^n = 1^n = 1$, which means one solution $(x + y\sqrt{d}) \longrightarrow (x, y)$ can generate *infinitely* many other solutions: $(x + y\sqrt{d})^n \longrightarrow (x_n, y_n)$ [5]. However, this does not mean that *all* solutions can be generated. Nor does it mean that one nontrivial solution always exists. These are not obvious, but for the moment proofs for them will be omitted.

We say a solution pair (x_0, y_0) is a fundamental solution if x and y are the smallest positive values. How can a fundamental solution be found for a non-square d value?

The equation $x^2 - dy^2 = 1$ can be written as

$$\left| \frac{x}{y} - \sqrt{d} \right| = \frac{1}{y|x + y\sqrt{d}|}. \tag{5}$$

For large positive integers x and y, the fraction x/y is a convergent to $\sqrt{d}$. So, to examine rational approximation methods for $\sqrt{d}$ is an alternative. The most popular method is to apply continued fractions but here we introduce another algorithm using intermediate fractions because of its relevance to the algorithm described in section 4.3.3.

Let p, q, r, s be natural numbers and z a real non-rational number. If $\frac{p}{q} < \frac{r}{s}$, then $\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}$.

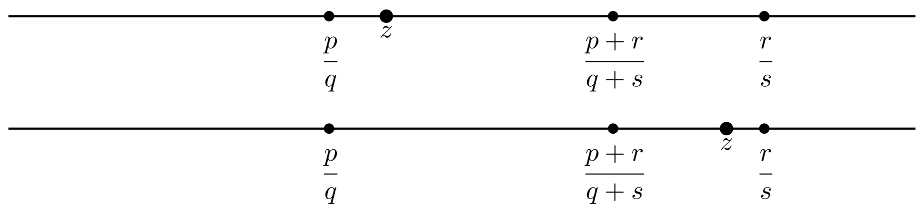
Proof

$$\frac{p}{q} < \frac{r}{s} \Longleftrightarrow ps - qr < 0$$

$$\frac{p}{q} - \frac{p+r}{q+s} = \frac{p(q+s) - q(p+r)}{q(q+s)} = \frac{ps - qr}{q(q+s)} < 0.$$

So, $\frac{p}{q} < \frac{p+r}{q+s}$. We apply a similar argument for the inequality $\frac{p+r}{q+s} < \frac{r}{s}$. ■

If a real number z lies within the interval $[\frac{p}{q}, \frac{r}{s}]$, then z must belong to either $[\frac{p}{q}, \frac{p+q}{r+s}]$ or $[\frac{p+q}{r+s}, \frac{r}{s}]$. This can be illustrated on a real number line.



In short, a procedure for approximating z is:

1. Choose $\frac{p}{q}$ and $\frac{r}{s}$ for which z lies within the interval $[\frac{p}{q}, \frac{r}{s}]$.
2. Add numerators and denominators of these fractions $\frac{p+r}{q+s}$.
3. Decide which of the smaller intervals $[\frac{p}{q}, \frac{p+r}{q+s}]$ and $[\frac{p+r}{q+s}, \frac{r}{s}]$ z lies within.
4. Repeat step 2 and 3 for the new interval.

If $z^2 = d$ for some integer d, we use the expression $x^2 - dy^2$ to decide which interval z belongs to. If $\frac{x}{y} < z$, then $x^2 - dy^2 < 0$. If $\frac{x}{y} > z$, then $x^2 - dy^2 > 0$.

$$\frac{x}{y} < \sqrt{d} \Longleftrightarrow \left(\frac{x}{y}\right)^2 - d < 0 \Longleftrightarrow x^2 - dy^2 < 0$$

Here, x and y are positive integral values. For example, the approximation of $\sqrt{13}$ is the following sequence:

$$\frac{3}{1}\ ,\ \frac{4}{1}\ ,\ \frac{7}{2}\ ,\ \frac{11}{3}\ ,\ \frac{18}{5}\ ,\ \frac{29}{8}\ ,\ \ldots$$

This sequence is not unique. The following table will exemplify this algorithm for $d = 13$:

Table 1: The Intermediate Fraction Algorithm at work

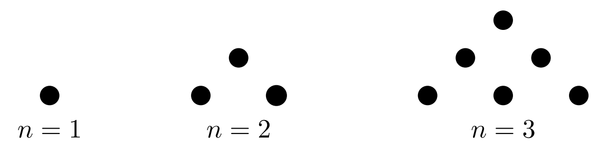
x	y	$x^2 - 13y^2$
3	1	−4
4	1	3
7	2	−3
11	3	4
18	5	−1

After ten more steps, we find the fundamental solution $(x_0, y_0) = (649, 180)$ to $x^2 - 13y^2 = 1$. This method always works because of its similarities to the proof of the intermediate value theorem [7].

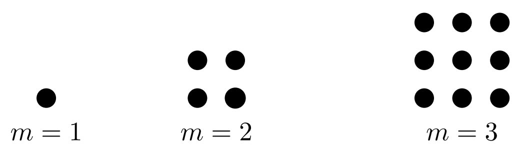
2.3 Two Problems Involving Pell's Equations

2.3.1 Square and Triangular Numbers

This is a classic example in which solving the Pell's equation is necessary. Triangular numbers have the form $\frac{1}{2}n(n+1)$ for some integer n.



A square is a number of the form m^2 for some integer m.



We want to know which triangular number is also a square. This leads to the equation $\frac{1}{2}n(n+1) = m^2$, which can be manipulated to the form:

$$\begin{aligned} \frac{1}{2}n(n+1) = m^2 &\Longleftrightarrow \frac{8}{2}n(n+1) = 8m^2 \\ &\Longleftrightarrow 4n^2 + 4n = 8m^2 \\ &\Longleftrightarrow (2n+1)^2 - 1 = 8m^2 \\ &\Longleftrightarrow (2n+1)^2 - 8m^2 = 1. \end{aligned}$$

Let $x = 2n+1$ and $y = m$. We apply the intermediate fraction method and find the fundamental solution $(x, y) = (3, 1)$ to $x^2 - 8y^2 = 1$. From this, all solutions can be derived by the operation $(3,1)^n \longrightarrow (3 + 1 \cdot \sqrt{8})^n$, for integer n. For example $(3+\sqrt{8})^2 = 9 + 6\sqrt{8} + 8 = (9+8) + 6\sqrt{8} = 17 + 6\sqrt{8}$, which gives the solution $(17, 6)$. Some solutions can be found in table 2.3.1. Another way to solve this problem is to find solutions to the equation $x^2 - 2y^2 = 1$.

Table 2: Some Solutions for the Square Triangular Numbers Problem

$(x, y) = (2n+1, m)$	(n, m)
$(3, 1)$	$(1, 1)$
$(17, 6)$	$(8, 6)$
$(99, 35)$	$(49, 35)$
$(577, 204)$	$(288, 204)$
$(3363, 1189)$	$(1161, 1189)$

2.3.2 Archimedes' Cattle Problem

In this section, "supreme wisdom" is not to be achieved. Only the connection between Archimedes' Cattle Problem and Pell's equation will be made clear. For a more rigorous treatment of this problem, pieces of literature like *Solving the Pell Equation* by H.W.Lenstra.Jr. or *Das Problema bovinum des Archimedes* by A.Amthor and B.Krumbiegel are recommended.

The problem is written in an epigram, describing the herd of the Sun God. There are two kinds of cattle: bulls and cows. The bulls can be divided into groups of white, black, yellow and dappled bulls. Let the number of bulls in each group be X, Y, Z and W respectively. The cows consist of white, black, dappled and yellow cows. Let the number of cows of each kind be x, y, z, and w. The first part involves solving the following system of equations:

$$\begin{aligned} X &= \left(\frac{1}{2}+\frac{1}{3}\right)Y + W, & x &= \left(\frac{1}{3}+\frac{1}{4}\right)(Y+y),\\ Y &= \left(\frac{1}{4}+\frac{1}{5}\right)Z + W, & y &= \left(\frac{1}{4}+\frac{1}{5}\right)(Z+z),\\ Z &= \left(\frac{1}{6}+\frac{1}{7}\right)X + W, & z &= \left(\frac{1}{5}+\frac{1}{6}\right)(W+w),\\ & & w &= \left(\frac{1}{6}+\frac{1}{7}\right)(X+x). \end{aligned}$$

This system has the following set of solutions, where u is a positive integer parameter:

$$\begin{aligned} X &= 10366482u, & x &= 7206360u,\\ Y &= 7460514u, & y &= 4893246u,\\ Z &= 7350860u, & z &= 3515820u,\\ W &= 4149387u, & w &= 5439213u. \end{aligned}$$

One who is able to solve this part is merely called "wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise" [4]. The "supreme wisdom" lies within the second part, which requires:

$$X + Y = m^2 \tag{6}$$

$$Z + W = \frac{1}{2}n(n+1), \quad m, n \in \mathbb{Z}^+ \tag{7}$$

In the condition (6), $X + Y = 1782699u = 2^2 \cdot 3 \cdot 11 \cdot 29 \cdot 4657u$, which is a square if, and only if, u has the form $3 \cdot 11 \cdot 29 \cdot 4657 \cdot t^2$. Together with the requirement (7), we find Pell's equation:

$$\Longleftrightarrow (2m+1)^2 - 410286423278424t^2 = 1.$$

2.4 Published Papers Related to the Cubic Pell's Equation

There have been attempts to deal with this equation. The oldest material is probably E. Meissel's paper, *Betrag zur Pell'scher Gleichung höheher Grad*. This was published in 1891. Others include: *On the Arithmetic Theory of the Form* $x^3 + ny^3 + n^2z^3 - 3nxyz$ (1980) by G.B.Mathews and *Normal Ternary Continued Fraction Expansions for Cubic Irrationals* (1929) by P.H.Daus and *The Theory of Irrationalities of the Third Degree* (1964) by B.N.Delone and D.K.Faddeev.

The Russian mathematician T.W.Cusick has written numerous papers treating fundamental units in the cubic fields. Together with Lowell Schoenfeld, they made a table of fundamental pairs of units in totally real cubic fields introduced in *Mathematics of Computation* (1987).

3 The Equation $x^3 - dy^3 = 1$

The first case to look at is when $d = r^3$ for some integer r.

$$\begin{aligned}
x^3 - r^3y^3 =& (x - ry)(x^2 + rxy + r^2y^2) \\
=& (x - ry)\frac{1}{2}(2x^2 + 2rxy + 2r^2y^2) \\
=& (x - ry)\frac{1}{2}(x^2 + (x^2 + 2rxy + r^2y^2) + r^2y^2) \\
=& (x - ry)\frac{1}{2}(x^2 + (x + ry)^2 + r^2y^2)
\end{aligned}$$

Since the factor $x^2 + (x+ry)^2 + r^2y^2$ is positive and both factors are integers, any solutions of $x^3 - dy^3 = \pm 1$ must satisfy $x^2 + rxy + r^2y^2 = 1$, which is equivalent to

$$\Longleftrightarrow x^2 + (x + ry)^2 + r^2y^2 = 2.$$

The terms $x^2, (x + ry)^2$ and $(ry)^2$ are non-negative integers. Consequently, one term must vanish. For $|r| > 1 \Rightarrow r^2 \geqslant 4$, the vanished term must be r^2y^2 so $y = 0$. Solutions satisfying this equation are $(1, 0)$ and $(-1, 0)$. For $|r| = 1$, we have $x^2 + (x + y)^2 + y^2 = 2$ or $x^2 + (x - y)^2 + y^2 = 2$. If $x = 0$, then $y = \pm 1$. If $y = 0$, we have $x = \pm 1$. If the term $x - y = 0$ and $x = y$, we can find a solution $(1, 1)$ for the equation $x^2 + (x - y)^2 + y^2 = 2$. For the case $x + y = 0$, we obtain the solutions $(1, -1)$ and $(-1, 1)$. However, these three solutions $(1, 1), (1, -1)$ and $(-1, 1)$ do not satisfy the equation $x^3 \pm y^3 = 1$. In conclusion, the only solutions for the case $|r| = 1$ are $(\pm 1, 0)$ and $(0, \pm 1)$. For $|r| > 1$, there exist only trivial solution pairs $(\pm 1, 0)$.

The equation $x^3 - dy^3 = 1$ is symmetric. Let $(x, y) = (x_1, y_1)$ be a solution of $x^3 - dy^3 = 1$. Then $(-x_1, -y_1)$ is a solution of $x^3 - dy^3 = -1$.

$$(-x_1)^3 - d(-y_1)^3 = -x_1^3 + dy_1^3 = -(x_1^3 - dy_1^3) = -1.$$

Another symmetry is that (x_1, y_1) is a solution pair to $x^2 - dy^2 = 1$ if, and only if, $(x_1, -y_1)$ satisfies $x^3 + dy^3 = 1$, which means investigating this equation for negative d values is not necessary.

One method to solve this equation for some non-cubic d values is to fix $y = 1, 2$ and 3. We will be able to examine conditions for d values. When $y = 1$,

$$x^3 - d = 1 \Leftrightarrow d = x^3 - 1 \tag{8}$$

So, if $d = s^3 - 1$ for some integer s, there always exists a solution (x, y), where $xy \neq 0$. For $y = 2$,

$$\begin{aligned}
& x^3 - 8d = 1 \\
& \Leftrightarrow x^3 = 8d + 1 \\
& \Leftrightarrow x \equiv 1 \pmod 8 \\
& \Leftrightarrow x = 8u + 1 \\
& \Leftrightarrow x^3 = (8u + 1)^3 = 1 + 8u(64u^2 + 24u + 3) \\
& \Leftrightarrow d = u(64u^2 + 24u + 3).
\end{aligned}$$

In a similar way, for all $y \neq 3n, n \in \mathbb{Z}$, an expression for d would be $u(y^6u^2 + 3y^3u + 3)$. When $y = 3$, it is more desirable to cube-complete $1 + 27d$, so that the smallest coefficients can be obtained.

$$\begin{aligned}(1 + 9u)^3 &= 1 + 9^3u^3 + 3 \cdot 9^2u^2 + 3 \cdot 9u \\ &= 1 + 27(27u^3 + 9u^2 + u)\end{aligned}$$

So, $d = u(27u^2 + 9u + 1)$. Generalizing a formula for d when $y = 3n$, we arrive at the expression $d = u(9n^6u^2 + 3m^3u + 1)$. Using this method and the symmetry of this equation, here are some $d \in [1, 200]$ and corresponding solutions.

Table 3: Solutions to $x^3 - dy^3 = 1$ for some d values

d	$d(u),\ u \in \mathbb{Z}$	$(x, y) : xy \neq 0$
2	$-[(-1)^3 - 1]$	$(-1, -1)$
7	$2^3 - 1$	$(2, 1)$
9	$-[(-2)^3 - 1]$	$(-2, -2)$
19	$-[(-1)(27 \cdot (-1)^2 + 9 \cdot (-1) + 1)]$	$(-8, -3)$
26	$3^3 - 1$	$(3, 1)$
28	$-[(-3)^3 - 1]$	$(-3, -1)$
37	$27 \cdot 1 + 9 \cdot 1 + 1$	$(10, 3)$
43	$-[(-1)(64 \cdot (-1)^2 + 24 \cdot (-1) + 3)]$	$(-7, -2)$
63	$4^3 - 1$	$(4, 1)$
65	$-[(-4)^3 - 1]$	$(-4, -1)$
91	$64 \cdot 1 + 24 \cdot 1 + 3$	$(9, 2)$
124	$5^3 - 1$	$(5, 1)$
126	$-[(-5)^3 - 1]$	$(-5, -1)$
182	$-[(-2)(27 \cdot (-2)^2 + 9 \cdot (-2) + 3)]$	$(-17, -3)$

Are these *all* possible solutions that can be found in this method? I have checked all d values in the interval $[1, 50]$ with Wolfram Mathematica Student version 7. It turns out that this method has omitted two d values and their corresponding solutions, which are $(18, 7)$ for $d = 17$ and $(-19, -7)$ for $d = 20$. For some d, the computational time in Mathematica is longer and the larger d grows, the more time it takes. Here is the list of troublesome d values: 23, 29, 30, 31, 34, 38, 41, 42, 45, 46, 47 and 50.

Obviously, this method is not useful for larger y values. A possibility is to consider $x^3 \equiv \pm 1 \pmod{y}$.

4 The Equation $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$

4.1 Definition

In this section, we define a version of the cubic Pell's equation analogously to the definition of the quadratic case.

Let c be a non-perfect cube, $c \in \mathbb{Z}$. The equation $t^3 - c = 0$ has the roots $\theta, \theta\varphi$ and $\theta\varphi^2$, where θ is real and $\varphi = \frac{1}{2}(-1 + \sqrt{-3})$. We consider the numbers of the form $x + y\theta + z\theta^2$ where x, y and z are rational numbers. We gather all real numbers of the form $x + y\theta + z\theta^2$ in a set $\mathbb{Q}(\theta)$. In other words, the definition of this set is:

$$\mathbb{Q}(\theta) = \{\alpha = x + y\theta + z\theta^2 : x, y, z \in \mathbb{Q}\}.$$

Here, $\mathbb{Q}(\theta)$ is a field. Take any two elements in a field and perform addition (subtraction), multiplication and division, the result is a real number that also belongs to $\mathbb{Q}(\theta)$. Next, we define the norm function in this field.

Definition 1 *The norm N_θ is a function on $\mathbb{Q}(\theta)$ which for every $\alpha = x + y\theta + z\theta^2$ is defined as $N(\alpha) = (x+y\theta+z\theta^2)(x+y\varphi\theta+z\varphi^2\theta^2)(x+y\varphi^2\theta+z(\theta\varphi^2)^2)$, where $x, y, z \in \mathbb{Q}$* [8].

Naturally, the norm is always rational if x, y and z are rational. This will be made clear later when we simplify the norm and examine it. It is required that x, y and z are integers. A more important subset is therefore:

$$\mathbb{Z}(\theta) = \{\alpha = x + y\theta + z\theta^2 : x, y, z \in \mathbb{Z}\}.$$

The subset $\mathbb{Z}(\theta)$ is a ring, i.e. the sum (the difference) and the product of two elements is another element belonging to $\mathbb{Z}(\theta)$. Solutions of this equation are nontrivial units in $\mathbb{Z}(\theta)$. The norm of a number in this ring is an integer.

Definition 2 *A unit is a number $\alpha = x + y\theta + z\theta^2$ in $\mathbb{Z}(\theta)$ such that its reciprocal $\dfrac{1}{\alpha}$ also belongs to $\mathbb{Z}(\theta)$. Denote a unit ϵ. A unit is trivial if $xyz = 0$.*

To prove that solutions to $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$ are nontrivial units in $\mathbb{Z}(\theta)$, one can first show that $\frac{N(\alpha)}{\alpha}$ is a member in $\mathbb{Z}(\theta)$. Next, one has to prove that if the norm equals 1, $x + y\theta + z\theta^2$ is a unit and a solution. To argue for the validity of the converse, one can use **Corollary 1.1** in the section *The Norm*.

4.2 Preliminary Examinations

4.2.1 The Norm

An investigation of the norm will give us some hints about this equation. The norm in **Definition 1** is not simplified. To do this, we verify that $\varphi^3 = 1$ and $\varphi^2 + \varphi + 1 = 0$. We will also use the reduction $\theta^3 = c$. The expression $(x + y\varphi\theta + z\varphi^2\theta^2)(x + y\varphi^2\theta + y(\theta\varphi^2)^2)$ can be shortened as:

$$
\begin{aligned}
&(x + y\varphi\theta + z\varphi^2\theta^2)(x + y\varphi^2\theta + y(\theta\varphi^2)^2) \\
&= x^2 + z^2c\theta + y^2\theta^2 + xy\theta\varphi^2 + xy\theta\varphi + xz\theta^2\varphi^2 + xz\theta^2\varphi + yzc\varphi^2 + yzc\varphi^2 \\
&= x^2 + z^2c\theta + y^2\theta^2 + xy\theta(\varphi^2 + \varphi + 1) + xz\theta^2(\varphi^2 + \varphi + 1) + yzc(\varphi^2 + \varphi + 1) - xy\theta - xz\theta^2 - yzc \\
&= x^2 + z^2c\theta + y^2\theta^2 - xy\theta - xz\theta^2 - yzc \\
&= (x^2 - yzc) + (z^2c - xy)\theta + (y^2 - xz)\theta^2.
\end{aligned}
$$

Next, we rewrite the norm of $x + y\theta + z\theta^2$ as:

$$
\begin{aligned}
N(x + y\theta + z\theta^2) &= (x + y\theta + z\theta^2)[(x^2 - yzc) + (z^2c - xy)\theta + (y^2 - xz)\theta^2] \\
&= x^3 + cy^3 + c^2z^3 - 3cxzy.
\end{aligned}
$$

Needless to say, the norm of $1 + 0 \cdot \theta + 0 \cdot \theta^2$ is 1. We want to examine the norm of numbers in the set $\mathbb{Z}(\theta)$. Let $\alpha = x + y\theta + z\theta^2$ and $\beta = u + v\theta + w\theta^2$:

Theorem 1 *The norm is multiplicative, i.e.* $N(\alpha\beta) = N(\alpha) \cdot N(\beta)$.

Proof This is a long proof for something simple.

$\alpha \cdot \beta = (x+y\theta+z\theta^2)(u+v\theta+w\theta^2) = (xu+ywc+zvc)+(xv+yu+zwc)\theta+(xw+yv+zu)\theta^2.$

The norm of this according to **Definition 1** is

$$
\begin{aligned}
&(xu + ywc + zvc)^3 + c(xv + yu + zwc)^3 + c^2(xw + yv + zu)^3 \qquad (9) \\
&-3c(xu + ywc + zvc)(xv + yu + zwc)(xw + yv + zu)
\end{aligned}
$$

We expand, distribute and simplify. The following expression will be obtained:

$$
\begin{aligned}
&(xu)^3 + (cyw)^3 + (czv)^3 + c(xv)^3 + c(yu)^3 + \qquad (10) \\
&c^4(zwc)^3 + c^2(xw)^3 + c^2(yv)^3 + c^2(zu)^3 + 9c^2xyzuvw \\
&-3c(x^3uvw + u^3xyz + cy^3uvw + cv^3xyz + c^2z^3uwc + c^2w^3xyz.)
\end{aligned}
$$

This is verified on Wolfram Mathematica. On the other hand, we have:

$$N(\alpha) \cdot N(\beta) = (x^3 + cy^3 + c^2z^3 - 3cxyz)(u^3 + cv^3 + c^2w^3 - 3cuvw)$$

We distribute, rearrange and compare the terms to (10).

$$\begin{gathered}(xu)^3 + c(yu)^3 + c^2(zu)^3 + c(xv)^3 + c^2(yv)^3 + c^3(zv)^3 \\ +c^2(xw)^3 + c^3(yw)^3 + c^4(zw)^3 + 9c^2xyzuvw \\ -3c(x^3uvw + u^3xyz + cy^3uvw + cv^3xyz + c^2z^3uvw + c^2w^3xyz).\end{gathered} \tag{11}$$

Evidently, the difference of (10) and (11) is 0, i.e. $N(\alpha\beta) = N(\alpha)N(\beta)$. ■

Corollary 1

1. $$N\left(\frac{\beta}{\alpha}\right) = \frac{N(\beta)}{N(\alpha)}, \quad N(\alpha) \neq 0.$$

2. $$N((\alpha)^n) = N(\alpha)^n, n \in \mathbb{Z}.$$

Proof

1. $$N(\beta) = N\left(\alpha \cdot \frac{\beta}{\alpha}\right) = N(\alpha)\ N\left(\frac{\beta}{\alpha}\right)$$
$$\iff \frac{N(\beta)}{N(\alpha)} = N\left(\frac{\beta}{\alpha}\right).$$

2. For $n = 2$, we substitute β with α in **Theorem 1** and get $N(\alpha^2) = [N(\alpha)]^2$. Suppose that $N(\alpha^n) = N(\alpha)^n$ is true for positive n. Then for $n+1$, we have the norm $N(\alpha^{n+1}) = N(\alpha \cdot \alpha^n) = N(\alpha)N(\alpha)^n = [N(\alpha)]^{n+1}$. So, this statement is true for $n = 2$ and if it is true for n, it is also true for $n + 1$. By induction, this is true for all positive n.

   For negative n, we simply combine **Corollary 1.1** and this statement for positive n:

$$\begin{aligned}&N(\alpha^{-n}) = N\left(\frac{1}{\alpha^n}\right) \\ &= \frac{1}{N(\alpha)^n} = \frac{1}{N(\alpha)^n} \\ &= \frac{1}{[N(\alpha)]^n} = [N(\alpha)]^{-n}. \ ■\end{aligned}$$

One observation can be made: if there exists one solution for a certain non-cubic integer c, an infinitude of other solutions can be found. Suppose $(x_0 + y_0\theta + z_0\theta^2)$ is a number that satisfies $N(x + y\theta + z\theta^2) = 1$, then other solutions can be generated by taking the power of this number. For the power $n = 2$, we expand and then use the identity $\theta^3 = c$ to simplify.

$$(x_0 + y_0\theta + z_0\theta^2)^2 = [(x_0^2 + 2cy_0z_0) + (2x_0y_0 + cz_0^2)\theta + (2x_0z_0 + y_0^2)\theta^2].$$

So, a new solution triple for the equation $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$ is $(x_0^2 + 2cy_0z_0, 2x_0y_0 + cz_0^2, 2x_0z_0 + y_0^2)$.

We exemplify with a case. A solution for $x^3 + 2y^3 + 4z^3 - 6xyz = 1$ is the triple $(1, 1, 1)$.

$$\begin{aligned}
(1,1,1) \quad & \longrightarrow \quad (1+\theta+\theta^2) \\
(1,1,1)^2 \quad & \longrightarrow \quad (1+\theta+\theta^2)^2 \\
& = (5+4\theta+3\theta^2) \longrightarrow (5,4,3).
\end{aligned}$$

We verify: $5^3 + 2 \cdot 4^3 + 4 \cdot 3^3 - 3 \cdot 2 \cdot 5 \cdot 4 \cdot 3 = 125 + 128 + 108 - 360 = 361 - 360 = 1.$

On the other hand, this does not mean *all* solutions can be found by taking this action. Neither does it imply that for all c, there exists a nontrivial solution.

4.2.2 Solutions for Negative c Values

Like $x^3 - dy^3 = 1$, this equation is also symmetric.

Theorem 2 *If $c = -d$, (u, v, w) is a solution of $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$ if and only if $(u, -v, w)$ is a solution of $x^3 + dy^3 + d^2z^3 - 3dxyz = 1$.*

Proof (u, v, w) is a solution of $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$ if and only if

$$\begin{aligned}
& u^3 + cv^3 + c^2z^3 - 3cuvw \\
& = u^3 + (-c)(-v)^3 + (-c)^2z^3 - 3(-c)u(-v)z \\
& = u^3 + d(-v) + d^2z^3 - 3du(-v)z = 1. \blacksquare
\end{aligned}$$

So, solutions of $x^3 - cy^3 + c^2z^3 + 3cxyz = 1$ are easily obtained from solutions of $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$. In other words, investigating positive c values is sufficient.

4.2.3 When c is a Perfect Cube

The question to be answered here is whether nontrivial solution triples exist when c is a perfect cube, i.e. $c = a^3$ for some positive integer a. We manipulate $x^3+cy^3+c^2z^3-3cxyz = 1$ to obtain the following expression:

$$\begin{aligned} x^3 + a^3y^3 + a^6z^3 - 3a^3xyz &= (x + ay + a^2z)[(x^2 - yza^3) + (z^2a^3 - xy)\theta + (y^2 - xz)\theta^2)] \\ &= \frac{1}{2}(x + ay + a^2z)[(x - ay)^2 + a^2(y - az)^2 + (a^2z - x)^2]. \end{aligned}$$

Suppose (x, y, z) is a solution triple of the equation $x^3 + a^3y^3 + a^6z^3 - 3a^3xyz = 1$, then:

$$\begin{aligned} 1 &= \frac{1}{2}(x + ay + a^2z)[(x - ay)^2 + a^2(y - az)^2 + (a^2z - x)^2] \\ \Longleftrightarrow 2 &= (x + ay + a^2z)[(x - ay)^2 + a^2(y - az)^2 + (a^2z - x)^2]. \end{aligned}$$

The factor $(x - ay)^2 + a^2(y - az)^2 + (a^2z - x)^2$ is positive and consequently it must have the value 2. The terms $(x - ay)^2, a^2(y - az)^2$ and $(a^2z - x)^2$ are non-negative integers. Consequently, one term must disappear.

When $a > 1 \Rightarrow a^2 \geqslant 4$, the vanishing term must be $a^2(y - az)^2$ so that $y = az$. We insert $y = az$ in $(a^2z - x)^2$ and get $(ay - x)^2$. The factor has the expression

$$2(x - ay)^2 = 2 \Longleftrightarrow x = \pm 1 + ay.$$

When $x = 1+ay$, we insert this identify of x in $x+ay+a^2z$ and get $2ay+a^2z = 0$. Now, we have $y = az$ and $2y = az$. Clearly, these identities are satisfied if, and only if, $y = 0$. Then z also equals 0 and x must be 1. The only solution of $x^3+a^3y^3+a^6z^3-3a^3xyz = 1$ when $a > 1$ is $(x, y, z) = (1, 0, 0)$. When $x = ay-1$, an expression for y is $y = \frac{1}{a} - \frac{az}{2}$, which also leads us to the only possibility $(x, y, z) = (1, 0, 0)$.

What can be said about the case $a = 1$?

$$(x - ay)^2 + a^2(y - az)^2 + (a^2z - x)^2 = (x - y)^2 + (y - z)^2 + (z - x)^2 = 2.$$

One of the term $x - y$, $y - z$ and $z - x$ must disappear. Suppose $(x - y)^2 = 0$. Then $x = y$ and we have $2(y - z)^2 = 2$. Two identities of z are $y = z + 1$ or $y = 1 - z$. We insert $x = y$ in the factor $x + y + z$. This factor equals 1, so $2y + z = 1$. Another identity of y is $y = (1 - z)/2$. The only integral y satisfying all identities is $y = 0$, so a solution triple satisfying $x^3 + y^3 + z^3 - 3xyz = 1$ is $(0, 0, 1)$. Similar for other cases, two other solutions are $(1, 0, 0)$ and $(0, 1, 0)$ for $a = 1$.

To wrap it up, the equation $x^3 + a^3y^3 + a^6z^3 - 3a^3xyz = 1$, where a is any integer, only has trivial solutions.

4.3 Solving the Cubic Version of Pell's Equation - Some Strategies

4.3.1 Fixing $x = 1$

When $x = 1$, the equation $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$ becomes $cy^3 + c^2z^3 - 3cyz = 0$, which can be simplified further by dividing both sides with positive c values. The equation $y^3 + cz^3 - 3yz = 0$ is obtained.

$$\begin{aligned} y^3 + cz^3 - 3yz = 0 &\iff cz^3 = 3yz - y^3 \\ z \neq 0 &\iff c = \frac{3yz - y^3}{z^3} \\ &= \frac{y(3z - 3y^2)}{z^3}. \end{aligned} \tag{12}$$

Now, we fix $z = \pm 1, \pm 2, \pm 3$ and vary integer y values for each fixed z integer. Using the symmetry property in **Theorem 2**, we can find solutions to several $c \in [2, 300]$. These are listed in the following table.

Table 4: Some Solutions to $y^3 + cz^3 - 3yz = 0$

c	$(1,y,z)$	c	$(1,y,z)$	c	$(1,y,z)$	c	$(1,y,z)$
2	$(1,1,1)$	24	$(1,-9,3)$	68	$(1,12,-3)$	140	$(1,5,-1)$
4	$(1,1,-1)$	30	$(1,9,-3)$	70	$(1,8,-2)$	198	$(1,-6,1)$
5	$(1,4,-2)$	36	$(1,3,-1)$	76	$(1,4,-1)$	207	$(1,-12,2)$
6	$(1,-6,3)$	52	$(1,-4,1)$	110	$(1,-5,1)$	210	$(1,-18,3)$
10	$(1,6,-3)$	58	$(1,-8,2)$	120	$(1,-15,3)$	222	$(1,18,-3)$
11	$(1,4,-2)$	60	$(1,-12,3)$	122	$(1,-5,5)$	225	$(1,12,-2)$
14	$(1,2,-2)$	61	$(1,-16,4)$	128	$(1,5,-5)$	234	$(1,6,-1)$
18	$(1,-3,1)$	67	$(1,16,-4)$	130	$(1,15,-3)$		

Playing around with the formula (12), we will land at the following identity:

$$c = \frac{3y}{z^2} - \frac{y^3}{z^3}. \tag{13}$$

This identity tells us that c can also have the form $r + k^3$, where $k^3 = -\frac{y^3}{z^3}$ and $r = \frac{3y}{z^2}$. Substitute $k = -\frac{y}{z}$ in $\frac{3y}{z^2}$, we obtain $r = -3k/z \Leftrightarrow rz = -3k$. Let $z = -s$ so that the minus sign is absorbed so the solutions will have the form $(1, ks, -s)$. First, we consider integer k values. Two specializations are:

1. Let $s = 3t$, so that $k = rt$. The solutions in this case have the form $(x, y, z) = (1, 3rt^2, -3t)$. c values in $[2, 300]$ that have solutions of this form are 2, 6, 7, 9, 10, 24, 26, 28, 30, 60, 62, 63, 65, 66, 68, 120, 124, 126, 130, 213, 214, 215, 217, 218 and 219. (See table A.1 in the Appendix.)

2. Let $r = 3t$, so that $k = st.(x, y, z) = (1, s^2t, -s)$. This can be applied for the following c values 2, 4, 5, 11, 14, 18, 24, 30, 36, 52, 58, 61, 67, 70, 76, 110, 122, 128, 140, 198, 207, 210, 213, 219, 222, 225 and 234. (See table A.2 in the Appendix.)

However, k does not have to be an integer because solution triples can be obtained from rational k as well. Let $k = \frac{p}{q}$, then

$$c = k^3 + r = \frac{p^3}{q^3} + r \Longleftrightarrow r = c - \frac{p^3}{q^3} = \frac{cq^3 - p^3}{q^3}.$$

We insert $k = \frac{p}{q}$ and the above expression for r in $rs = 3k$, we arrive at a formula for $s = \frac{3pq^2}{cq^3 - p^3}$. The solutions for this can be expressed as

$$(x, y, z) = \left(1, \frac{3p^2q}{cq^3 - p^3}, \frac{-3pq^2}{cq^3 - p^3}\right)$$

For $q = 2$:

$$(x, y, z) = (1, \frac{6p^2}{8c - p^3}, \frac{-12p}{8c - p^3}).$$

This tells us that $(8c - p^3)$ must be a factor in both $6p^2$ and $-12p$, which means $8c - p^3 \in A = \{\pm 1, \pm 2, \pm 3, \pm p, \pm 6, \pm 2p, \pm 3p\}$. In addition, we require that $2 \nmid p$ because if so, k is an integer. A similar action is taken for $q = 3, 4, \ldots$. Solutions found for rational k are represented in table A.3 in the Appendix.

4.3.2 When $c = r^2$ and $c = sr^3$

The following will show a strategy to generate a solution from certain c values. However, this strategy often results in large x, y, and z values. We examine the case when c is a perfect square and when c is a multiple of a cube.

Theorem 3 *The triple* (u, v, w) *is a solution of*

$$x^3 + cy^3 + c^2z^3 - 3cxyz = 1 \tag{14}$$

if and only if (u, rw, v) *satisfies*

$$x^3 + ry^3 + r^2z^3 - 3rxyz = 1, \tag{15}$$

for $c = r^2$.

> *Proof* First, we will prove that (u, v, w) as a solution of equation (14) will lead to the solution (u, rw, v) of equation (15). Later, we will prove the converse of this.

- If (u, v, w) is a solution of equation (14), then (u, v, w) must satisfy $u^3 + cv^3 + c^2w^3 - 3cuvw = 1$. Insert $c = r^2$. We will get $u^3 + r^2v^3 + r^4w^3 - 3r^2uvw = 1$. A rearrangement of the terms in this equation will lead to:

$$u^3 + r^2v^3 + r^4w^3 - 3r^2uvw = u^3 + r(rw)^3 + r^2v^3 - 3ruv(rw) = 1.$$

  So, (u, rw, v) is a solution of equation (15) if (u, v, w) satisfies equation (14).

- Suppose (u, rw, v) is a solution of equation (15), then $u^3 + r(rw)^3 + r^2v^3 + 3ruv(rw) = u^3 + r^2v^3 + r^4w^3 - 3r^2uvw = 1$. Replace $r^2 = c$. We arrive at $u^3 + cv^3 + c^2w^3 - 3cuvw = 1$. Obviously, (u, v, w) is a solution of (14) if (u, rw, v) is a solution of (15).

In conclusion, (u, v, w) satisfies $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$ if, and only if, (u, rw, v) is a solution of $x^3 + ry^3 + r^2z^3 - 3rxyz = 1$.

From a c value, we can find a solution for its square. However, one known solution (x, y, z) of a c value is not enough. According to **Theorem 3**, c must divide y as well. The following example will illustrate how this theorem can be applied.

Example Starting from $c = 5$ and $(x, y, z) = (1, -4, 2)$, we can try to find a solution for $c = 25$. From **Theorem 3**, we have learn that y value in the solution triple must be divisible by $c = 5$. It is clear that -4 does not. So, we transform the triple to a number in $\mathbb{Z}(\theta)$: $(1, -4, 2) \Longrightarrow 1 - 4\theta + 2\theta^2$. We generate other solutions and seek for the smallest positive n so that $(1 - 4\theta + 2\theta^2)^n$ and for which $5|y$.

$$\begin{aligned}
(1 - 4\theta + 2\theta^2)^2 &\Longrightarrow (-79, 12, 20) \\
(1 - 4\theta + 2\theta^2)^3 &\Longrightarrow (-359, 528, -186) \\
&\cdots \\
(1 - 4\theta + 2\theta^2)^5 &\Longrightarrow (70001, -64620, 54490)
\end{aligned}$$

The fifth power of this number gives the y value that matches the requirement. A solution of $x^3 + 25y^2 + 255z^3 - 75xyz = 1$ is $(70001, 54490, -12924)$.

Theorem 4 *(u, v, w) is a solution to $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$ where $c = sr^3$ if and only if (u, rv, r^2w) is a solution to $x^3 + sy^3 + s^2z^3 - 3sxyz = 1$ for $c = sr^3$.*

Proof for this is an imitation of the one for **Theorem 3**.

The result of this section can be found in tables B.1, B.2 and B.2. In table B.2, we have found a solution for $c = 144$, which is $(293500801, 37638510, -17864080)$. We can use **Theorem 3** to find a solution for $c = 12$. A solution for $c = 12$ is $(293500801, -214368960, 37638510)$.

4.3.3 An Algorithm that Does Not Always Work

In this section, we introduce an algorithm that works surprisingly often but does not always work. The theory applied is the intermediate fraction method, which is mentioned in section 2.2.2, "A Short Overview of the Quadratic Pell's Equation".

A useful factorization that has been introduced earlier is

$$x^3 + cy^3 + c^2z^3 - 3cxyz = \frac{1}{2}(x + y\theta + z\theta^2)[(x - y\theta)^2 + (y\theta - z\theta^2)^2 + (x - z\theta^2)^2].$$

For a solution triple that consists of large positive x, y and z values and θ is the real root of a noncubic positive integer c, the factor $x + y\theta + z\theta^2 \to \infty$ and consequently the other one approaches 0. This means the terms $(x - y\theta)^2$, $(y\theta - z\theta^2)^2$ and $(x - z\theta)^2$ will be close to 0. This property will help us to trace solutions to $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$.

1. Let p be the integral part of the real cube root of c, i.e. θ. For example, the integral part of $\sqrt[3]{9} = 2,08008...$ is $2 = p$. Let q be the integral part of $p\theta$. For example, $q = 4$ if $p\theta = 2\sqrt[3]{9}$. The first triple to begin with is $(x, y, z) = (q, p, 1)$. Plug this in $x - cy^3$ and $y^3 - cz^3$ and calculate. We are interested in which pair of signs among $(+, +)$, $(+, -)$, $(-, +)$ or $(-, -)$ the output has.

2. The next triple is $(q + 1, p, 1)$. Plug in and calculate. The third triple is $(r, p + 1, 1)$, where r is the integral part of $(p+1)\theta$. Do the same calculation with this as well.The following table illustrates the first 2 steps:

Table 5: The First Four Steps of the Algorithm

(x, y, z)	$x^3 - cy^3$	$y^3 - cz^3$
$(q, p, 1)$	$-$	$-$
$(q + 1, p, 1)$	$+$	$-$
$(r, p + 1, 1)$	$-$	$+$
$(r + 1, p + 1, 1)$	$+$	$+$

3. After n steps, we arrive at the nth triple, say (u, v, w). In the table,*the most recent* triple that yields the opposite signs to $u^3 - cv^3$ **and** $v^3 - cw^3$ are selected. Let this triple be (d, e, f). The $n + 1$-th triple is $(u + d, v + e, w + f)$. For example, the most recent triple that yields the opposite signs to $(r + 1)^3 - c(p + 1)^3$ and $(p + 1)^3 - c$ is $(q, p, 1)$. So, the fifth triple is $(r + q + 1, 2p + 1, 2)$. This step is iterated until a solution is found.

We exemplify this by finding solutions for $c = 2$. The integral part p of $\sqrt[3]{2}$ is 1. Since q is the integral part of $p\sqrt[3]{2} = 1 \cdot \sqrt[3]{2}$, $q = 1$. r is the integral part of $(p + 1)\sqrt[3]{2}$, $r = 2$. The first 19 steps are detailedly represented in the table 4.3.3.

Table 6: The Algorithm at Work for $c = 2$

(x, y, z)	$x^3 - 2y^3$	$y^3 - 2z^3$	$x^3 + 2y^3 + 4z^3 - 6xyz$
$(1, 1, 1)$	$-$	$-$	1
$(2, 1, 1)$	$+$	$-$	2
$(2, 2, 1)$	$-$	$+$	4
$(3, 2, 1)$	$+$	$+$	11
$(4, 3, 2)$	$+$	$+$	6
$(5, 4, 3)$	$-$	$+$	1
$(7, 5, 4)$	$+$	$-$	9
$(12, 9, 7)$	$+$	$+$	22
$(13, 10, 8)$	$+$	$-$	5
$(18, 14, 11)$	$+$	$+$	12
$(19, 15, 12)$	$+$	$-$	2
$(24, 19, 15)$	$+$	$+$	9
$(25, 20, 16)$	$-$	$-$	5
$(49, 39, 31)$	$-$	$-$	1
$(73, 58, 46)$	$-$	$+$	2
$(92, 73, 58)$	$+$	$-$	3
$(165, 131, 104)$	$-$	$-$	5
$(189, 150, 119)$	$+$	$+$	2
$(354, 281, 223)$	$-$	$+$	4
...	...	...	...

We find four solutions:

$$(1, 1, 1) = (1, 1, 1)^1$$
$$(5, 4, 3) = (1, 1, 1)^2$$
$$(19, 15, 12) = (1, 1, 1)^3$$
$$(73, 58, 46) = (1, 1, 1)^4$$

The solution $(1, 1, 1)^5 = (281, 223, 177)$ does not seem to appear in this table. This is enough proof of this algorithm's inability to generate all solutions. The solutions found in table C in the Appendix are computed in MatLab R2011a within 1000 iterations (`MAXn=1000`). It is necessary to be aware of that MatLab cannot calculate a small difference between two large values. In this case, it will not recognize the difference between $x^3 - cy^3$ and $y^3 - cz^3$ for most of d values larger than 100. So, the maximum number of iterations at 1000 is a reasonable limit with regards to this computational weakness. So far, the algorithm does not succeed in finding the smallest positive solution triples for $c =$ 15, 16, 17, 20, 23, 25, 32, 33, 34, 41, 42, 44, 45, 46, 56, 59, 69, 71, 72, 74, 75, 77–79, 80–85, 87, 89, 92 and 92–100.

5 Application of the Cubic Pell's Equation in Approximation Theory

In the last section, we have introduced an algorithm partly based on the approximation of positive noncubic integer c. Conversely, a solution (x, y, z) to the cubic version of Pell's equation with large positive x, y and z will give $\frac{x}{y}$ and $\frac{y}{z}$, which are close to the real cube root of c.

Instead of expanding $(x + y\theta + z\theta^2)^n$, here is a shortcut to obtain a solution triple with this property. Consider a solution $(x, y, z) \longrightarrow x + y\theta + z\theta^2$, the reciprocal of $x + y\theta + z\theta^2$ is

$$(x + y\theta + z\theta^2)^{-1} = \frac{1}{x + y\theta + z\theta^2} = \frac{N(x + y\theta + z\theta^2)}{x + y\theta + z\theta^2}.$$

When the norm is factorized, the number $x + y\theta + z\theta^2$ will disappear and

$$\begin{aligned}(x + y\theta + z\theta^2)^{-1} = \frac{1}{2}((x - y\theta)^2 + (y\theta - z\theta^2)^2 + (x - z\theta^2)^2) \\ = x^2 + y^2\theta^2 + cz^2\theta - (xy\theta + cyz + xz\theta^2).\end{aligned}$$

A rearrangement of terms gives a number $u + v\theta + w\theta^2$, where $u = x^2 - cyz$, $v = cz^2 - xy$ and $w = y^2 - xz$. The integers u, v and w are positive if at least one out of x, y or z is negative. For example, the inverse of the solution $(1, 100, -80)$ for $c = 2$ is

$$\begin{cases} u = x^2 - cyz = 1 - 2 \cdot 100 \cdot (-80) = 16001 \\ v = cz^2 - xy = 2 \cdot (-80)^2 - 1 \cdot 100 = 12700 \\ w = y^2 - xz = 100^2 - 1 \cdot (-80) = 10080 \end{cases}$$

A new solution triple is $(x, y, z) = (16001, 12700, 10080)$. Let us test these approximations of $\sqrt[3]{2}$.

$$\begin{aligned}\left|\frac{x}{y} - \sqrt[3]{2}\right| = \left|\frac{16001}{12700} - \sqrt[3]{2}\right| = \\ |1.259921126 - \sqrt[3]{2} \approx 0.0099993783 < 10^{-3} \\ \left|\frac{y}{z} - \sqrt[3]{2}\right| = \left|\frac{12700}{10080} - \sqrt[3]{2}\right| = \\ |1.2599206635 - \sqrt[3]{2} \approx 4.149423 \cdot 10^{-7} < 10^{-6}.\end{aligned}$$

A weakness of this method is that it is hard to control how close the approximation is without a reference value.

6 Proposals for Continued Future Work

We have seen that there are numerous solutions to the cubic analogue $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$ of Pell's equation. Yet, there remains a lot to be done and theorems to be formulated. Here is a short wish-list for work in the near future.

1. Prove that there always exists a fundamental solution for all positive non-cubic integer c values.

2. Prove that a fundamental solution generates all solutions.

3. In this project, c values such as $t^3 - c = 0$ are considered. A generalization of the cubic Pell's equation would be a c value such that

   $$t^3 + at^2 + bt + c = 0,$$

   where a and b are integers. The polynomial expression on the left side is irreducible. The equation has real roots, which are θ_1, θ_2 and θ_3. The norm is

   $$(x + y\theta_1 + z\theta_1^2)(x + y\theta_2 + z\theta_2^2)(x + y\theta_3 + z\theta_3^2). \tag{16}$$

4. Examine whether the continued fraction approach is a possible method for certain c values. Compare this with the intermediate fraction method.

These goals are reasonably achievable. To prove that there always exists one fundamental solution for all non-cubic c is to imitate Dirichlet's proof for $x^2 - dy^2 = 1$, in which Dirichlet has used the pigeohole principle. Proof for the generating of all solutions from one nontrivial solution requires more knowledge about units in the field $\mathbb{Q}(\theta)$.

These goals aside, I want to examine the equation $x^3 - dy^3 = 1$ more closely by considering its factorization in the field $\mathbb{Q}(\sqrt{-3})$ as I intended to do from the beginning. If we can prove for example $(-1, -1)$ is the only solution to $x^3 - 2y^3 = 1$, then we can prove that $d = 2r$ for some integer r does not have nontrivial solutions. In this project, I have not considered other approaches to the equation $x^3 - dy^3 = 1$, such as fixing d values instead of y values.

One step further is to look at Pell's equation of higher degrees. This time, it is not necessary to consider $x^n - dy^n = 1$ and one can define the Pell's equation of higher degrees directly as

$$\begin{aligned} g_c(x_1, x_2, x_3, ..., x_n) &= N(x_1 + x_2\theta + x_3\theta^2 + ... + x_n\theta^{n-1}) \\ &= \prod_{i=0}^{n} \left(x_1 + x_2(\xi^i\theta) + x_3(\xi^i\theta)^2 + ... + x_n(\xi^i\theta)^{n-1}\right) = 1, \end{aligned}$$

where $\xi = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ is a primitive nth root of unity [9].

A Results for Section 4.3.1.

A.1 Part 1: $k = rt$

$$c = k^3 + r$$
$$k = rt$$
$$(x, y, z) = (1, 3rt^3, -3t).$$

Table 7: Solutions for $c = r^3t^3 + r$

c	t	r	$(1, y, z)$	$(1, y, z)^{-1}$
2	1	1	$(1, 3, -3)$	$(19, 15, 12)$
6	-1	-2	$(1, -6, 3)$	$(109, 60, 33)$
7	-2	-1	$(1, -12, 6)$	$(505, 264, 138)$
9	2	1	$(1, 12, -6)$	$(649, 312, 150)$
10	1	2	$(1, 6, -3)$	$(181, 84, 39)$
24	-1	-3	$(1, -9, 3)$	$(649, 225, 78)$
26	-3	-1	$(1, -27, 9)$	$(6319, 2133, 720)$
28	3	1	$(1, 27, -9)$	$(6805, 2241, 738)$
30	1	3	$(1, 9, -3)$	$(811, 261, 84)$
60	-1	-4	$(1, -12, 3)$	$(2161, 552, 147)$
62	-2	-2	$(1, -24, 6)$	$(8929, 2256, 570)$
63	-4	1	$(1, -48, 12)$	$(36289, 9120, 2292)$
65	4	1	$(1, 48, -12)$	$(37441, 93112, 2316)$
66	2	2	$(1, 24, -6)$	$(9505, 2352, 582)$
68	1	4	$(1, 12, -3)$	$(2249, 600, 147)$
120	-1	-5	$(1, -15, 3)$	$(5401, 1095, 222)$
124	-5	-1	$(1, -75, 15)$	$(139501, 27975, 5610)$
126	5	1	$(1, 75, -15)$	$(141751, 28275, 5640)$
130	1	5	$(1, 15, -3)$	$(5851, 1155, 228)$
213	-2	-3	$(1, -36, 6)$	$(46009, 7704, 1290)$
214	-3	-3	$(1, -54, 9)$	$(104005, 17388, 2907)$
215	-6	-1	$(1, -108, 18)$	$(417961, 69768, 11646)$
217	6	1	$(1, 108, -18)$	$(421848, 70200, 11682)$
218	3	2	$(1, 54, -9)$	$(105949, 17604, 2925)$
219	2	3	$(1, 36, -6)$	$(47305, 7848, 1302)$

A.2 Part 2: $k = st$

$$
\begin{aligned}
c &= k^3 + r; \\
r &= 3t \\
k &= \frac{1}{3} rs = st \\
(x, y, z) &= (1, s^2 t, -s).
\end{aligned}
$$

Table 8: Solutions for $c = s^3 t^3 + 3t$

c	s	t	$(1, y, z)$	$(1, y, z)^{-1}$
2	−1	−1	(1, 1, 1)	(−1, 1, 0)
	−1	−2	(1, −2, 1)	(5, 4, 3)
4	1	1	(1, 1, −1)	(5, 3, 2)
5	−2	−1	(1, −4, 2)	(41, 24, 14)
11	2	1	(1, 4, −2)	(89, 40, 18)
14	1	2	(1, 2, −1)	(29, 12, 5)
18	−1	−3	(1, −3, 1)	(55, 21, 8)
24	−3	−1	(1, −9, 3)	(649, 225, 78)
30	3	1	(1, 9, −3)	(811, 261, 84)
36	1	3	(1, 3, −1)	(109, 33, 10)
52	−1	−4	(1, −4, 1)	(209, 56, 15)
58	−2	−2	(1, −8, 2)	(929, 240, 62)
61	−4	−1	(1, −16, 4)	(3905, 992, 252)
67	4	1	(1, 16, −4)	(4289, 1056, 260)
70	2	2	(1, 8, −2)	(1121, 272, 66)
76	1	4	(1, 4, −1)	(305, 72, 17)
110	−1	−5	(1, −5, 1)	(551, 115, 24)
122	−5	−1	(1, −25, 5)	(15251, 3075, 620)
128	5	1	(1, 25, −5)	(16001, 3175, 630)
140	1	5	(1, 5, −1)	(701, 135, 26)
198	−1	−6	(1, −6, 1)	(1189, 204, 35)
207	−2	−3	(1, −12, 2)	(4969, 840, 142)
210	−3	−2	(1, −18, 3)	(11341, 1908, 321)
213	−6	−1	(1, −36, 6)	(46009, 7704, 1290)
219	6	1	(1, 36, −6)	(47305, 7848, 1302)
222	3	2	(1, 18, −3)	(11987, 1980, 327)
225	2	3	(1, 12, −2)	(5401, 888, 146)
234	1	6	(1, 6, −1)	(1405, 228, 37)

A.3 Part 3: Rational k

$$c = k^3 + r$$
$$k = \frac{p}{q}$$
$$r = c - \frac{p^3}{q^3}$$
$$s = \frac{-3pq^2}{q^3 c - p^3}$$
$$(x, y, z) = \left(1, \frac{3p^2 q}{cq^3 - p^3}, \frac{-3pq^2}{cq^3 - p^3}\right)$$

Table 9: Solutions for $c = k^3 + r$, k is rational

c	p	q	$(1, y, z)$	$(1, y, z)^{-1}$
2	5	4	$(1, 100, -80)$	$(16001, 12700, 10080)$
3	3	2	$(1, -18, 12)$	$(649, 450, 312)$
4	8	5	$(1, 1280, -800)$	$(4096001, 2558720, 1639200)$
15	5	2	$(1, -30, 12)$	$(5401, 2190, 888)$
16	5	2	$(1, 50, -20)$	$(16001, 6350, 2520)$
19	2	3	$(1, 576, -216)$	$(2363905, 885888, 331992)$
22	14	5	$(1, 490, -175)$	$(1886501, 673260, 240275)$
37	10	3	$(1, -900, 270)$	$(89901001, 2698200, 809730)$
42	7	2	$(1, -42, 12)$	$(21169, 6090, 1752)$
43	7	2	$(1, 294, -84)$	$(1061929, 303114, 86520)$
91	9	2	$(1, -486, 108)$	$(4776409, 1061910, 236088)$
90	9	2	$(1, -54, 12)$	$(58321, 13014, 2904)$
152	16	3	$(1, 288, -54)$	$(2363905, 442944, 82998)$
165	11	2	$(1, -242, 44)$	$(1756921, 319682, 58520)$
166	11	2	$(1, -66, 12)$	$(131473, 23970, 4344)$
182	17	3	$(1, 2601, -459)$	$(217282339, 38341341, 6765660)$
254	19	3	$(1, -3249, 513)$	$(423351199, 66848175, 10555488)$
273	13	2	$(1, -78, 12)$	$(255529, 39390, 6072)$
275	13	2	$(1, 338, -52)$	$(3260401, 743262, 114296)$

B Results for Section 4.3.2

B.1 Part 1: $c = r^2$

$(x, y, z) = (u, v, w)$ is a solution of the equation $x^3 + cy^3 + c^2z^3 - 3cxyz = 1$.

$$(u, v, w)^c \longrightarrow (u_c, v_c, w_c), c|y_c$$

The solution of the equation $x^3 + c^2y^3 + c^4z^3 - 3c^2xyz = 1$ is

$$(u_c, w_c, \frac{v_c}{c})$$

Table 10: Solutions for Some Square c

c	(x, y, z)
16	$(1, 50, -20)$
49	$x = -24047260775$
	$y = -17296073382$
	$z = 6522442152$
81	$(-5831, 12078, -2480)$
100	$x = -25170395948999$
	$y = -812743134690$
	$z = 1343406488$
121	$x = -12514319138551$
	$y = 2641372166730$
	$z = -22487879872$
196	$x = 15812858692121$
	$y = -4263578789806$
	$z = 265347613029$
256	$x = 1868292792798067131617933568115504128000 1$
	$y = -476374665090531541384730236077960128 0$
	$z = 55166906770875419910219602713310390 0$

B.2 Part 2: $c = r^3 s$

Table 11: Solutions for nonsquare c

c	r	s	(x, y, z)
32	2	4	$(-31999, 4920, 1625)$
40	2	5	$(-79, 6, 5)$
48	2	6	$(67393, -3738, -4074)$
54	3	3	$(61561, 16287, 4309)$
56	2	7	$(-1007, 114, 39)$
72	2	9	$(-1295, 174, 33)$
80	2	10	$(190081, -32118, -2784)$
88	2	11	$(-175, 26, 3)$
108	3	4	$(-45359, -16053, 5371)$
112	2	14	$(4033, -962, 26)$
120	2	15	$(-10799, 1050, 231)$
128	4	2	$(-31999, 3250, 615)$
135	3	5	$(1586550241, -253319772, -10907294)$
152	2	19	$(-4727807, 443808, 82836)$
162	3	6	$(-971, 696, -95)$
176	2	22	$(21353248224001, -1913474927770, -338473111900)$
189	3	7	$(-4535, 6288, -958)$
192	4	3	$(9228015072001, -413477943660, -205600534275)$
208	2	26	$(85770034827984001, -6142575342403530, -1406482729723800)$
224	2	28	$(115391223262828801, -10885285834047720, -1336179005145135)$
240	2	30	$(22672264901761, -2608355136294, -167343454824)$
256	4	4	$(9228015072001, -205600534275, -413477943660)$
288	2	36	$(6003763201, -1070291040, 6003763201)$
296	2	37	$(-17981999, 1347750, 202635)$

Table 12: Solutions for Some Square c

c	r	s	(x, y, z)
81	3	3	$(2460229201, 568609218, 131417204)$
144	2	18	$(293500801, 37638510, -17864080)$
256	2	32	$(3534520707068416001, -251896104598970160, -47996066438088775)$

C Results for Section 4.3.3

See codes in D.2

Table 13: Solutions Computed by MatLab R2011a

c	(x, y, z)	c	(x, y, z)
2	$(1, 1, 1)$	37	$(100, 30, 9)$
3	$(4, 3, 2)$	38	$(29071, 8647, 2572)$
4	$(5, 3, 2)$	39	$(529, 156, 46)$
5	$(41, 24, 14)$	40	$(5041, 1474, 431)$
6	$(109, 60, 33)$	43	$(49, 14, 4)$
7	$(4, 2, 1)$	52	$(209, 56, 15)$
9	$(4, 2, 1)$	54	$(61561, 16287, 4309)$
10	$(181, 84, 39)$	58	$(929, 240, 62)$
11	$(89, 40, 18)$	60	$(2161, 552, 141)$
12	$(9073, 3963, 1731)$	61	$(3905, 992, 252)$
13	$(94, 40, 17)$	62	$(8929, 2256, 570)$
14	$(29, 12, 5)$	63	$(16, 4, 1)$
18	$(55, 21, 8)$	65	$(16, 4, 1)$
19	$(12304, 4611, 1728)$	66	$(9505, 2352, 582)$
21	$(1705, 618, 224)$	67	$(4289, 1056, 260)$
22	$(793, 283, 101)$	68	$(2449, 600, 147)$
24	$(649, 225, 78)$	70	$(1121, 272, 66)$
26	$(9, 3, 1)$	73	$(99928, 23910, 572)$
28	$(9, 3, 1)$	76	$(305, 72, 17)$
30	$(811, 261, 84)$	86	$(565, 128, 29)$
31	$(101209, 32218, 10256)$	88	$(2376, 5342, 1201)$
35	$(25776, 7880, 2409)$	91	$(81, 18, 4)$
36	$(109, 33, 10)$		

D Syntax and Codes

D.1 Syntax Used in Wolfram Mathematica

```
Reduce[x^2 - dy^2 == 1, {x, y}, Integers]
```
Solve a quadratic Pell's equation.

```
sol = Reduce[{x^2 - dy^2 == 1, x >0, y>0 }, {x, y}, Integers]
```
Give solution pairs (x, y) to a quadratic Pell's equation, x, y positive.

```
{x,y}\.First[sol]\.Table[{C[1] -> i}, {i,n}]\\Simplify
```
List n first solutions for e.g. the quadratic Pell's equation.

```
Reduce[x^3 - dy^3==1, {x,y}, Integers]
```
Solve the special case of a cubic Pell's equation

Alternatively: `Solve[x^2-dy^2 == 1, {x,y}, Integers]` also solves the quadratic Pell's equation.

```
Expand[(x + y\theta + z\theta^2)^n]
```
Expand. This command is used to verify Theorem 1 and to generate solutions in table B.1, B.2 and B.2.

```
Distribute[(x + y\theta + z\theta^2)*(u+v\theta + w\theta^2)]
```
Distribute terms to terms. This command is used to verify Theorem 1.

```
FullSimplify[(x + y\theta + z\theta^2)^(1/n)]
```
Simplify as much as possible.

D.2 Codes Used in MatLab

```
function xyz = algorithm1(c)
c = ('Insert a positive c value here')
MAXn=1000

x = zeros(MAXn, 1); y = zeros(MAXn, 1); z = zeros(MAXn, 1);
sign2 = zeros(MAXn, 2);

p0 = floor(sign(c)*abs(c)^(1/3);
q0 = floor(p0*sign(c)*abs(c)^(1/3);
r0 = floor((p0+1)*sign(c)*abs(c)^(1/3);


x(1)=q0; y(1)=p0; z(1)=1;
x(2)=q0+1; y(2)=p0; z(2)=1;
x(3)=r0; y(3)=p0+1; z(3)=1;
x(4)=r0+1; y(4)=p0+1; z(4)=1;

sol=0;
for n=1:4
sign2(n,:)  = [sign((x(n))^3-c*(y(n)^3), sign((y(n))^3 - c*(z(n))^3];
if (x(n)^3 + c*y(n)^3 + c^2*z(n)^3 - 3*c*x(n)*y(n)*z(n)) == 1
sol = 1;
break
end
end

if sol==0
for n = 5:MAXn
for m = n-2:-1:1
if max(abs(sign2(n-1,:)+sign2(m,:)))==0;
break
end
end

x(n)=x(n-1)+x(m); y(n)=y(n-1)+y(m); z(n)=z(n-1)+z(m);
sign2(n,:)=[sign((x(n))^3 - c*(y(n))^3), sign((y(n))^3 - c*(z(n))^3];
if (x(n)^3 + c*y(n)^3 + c^2*z(n)^3 - 3*c*x(n)*y(n)*z(n))==1
sol=1;
break
end
end
end

if sol == 1 xyz = [x(n), y(n), z(n)];
else
xyz=0;
end
```

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