Uniqueness of Solutions to the Poisson Equation

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Abstract

We show that the Poisson equation,

$$\begin{cases} \Delta u = f & \text{in } M, \\ u = g & \text{on } \partial M, \end{cases}$$

has at most one solution where M is open and bounded. As intermediate steps we also show two properties for harmonic functions. The first that for a harmonic function v, the value at any given point will equal the mean value of v in a ball centered around that point, and the second that the maximum and minimum value of any harmonic function can be found on the boundary ∂M of M.

1 Introduction

A common problem in mathematics is showing that an obtained solution is the only solution. This is often equally important as actually finding solutions since you have not proven anything unless you have taken everything in account.

Within the field of differential equations one often works with mathematical models for real life phenomena, for example electrical currents and heat conduction. These mentioned models include the so called *Laplace operator* Δ for which a function $u(x)$ of n variables is defined as the sum of all partial second derivates with respect to one variable,

$$\begin{cases} u & = u(x_1, x_2, \ldots, x_n) \\ \Delta u & = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \end{cases}$$

An equation based on Laplace's operator which has a broad use within the differential calculus field, among others in electrostatics and mechanical engineering, is *Poisson's equation*

$$\Delta u = f,$$

where f is a given function. What makes Poisson's equation interesting and relevant to study is its many applications.

What is investigated in this paper is if a solution to Poisson's equation is *unique*. Next it is explained how this is done.

A special case of Poisson's equation is when f equals 0. What is obtained is Laplace's equation

$$\Delta u = 0.$$

A function u which satisfies Laplace's equation is said to be harmonic [1]. In this paper it is first shown that a certain *mean value property* applies for harmonic functions. This in turn is used to prove the *maximum/minimum principle*, which says that for a harmonic function the minimum and maximum values are obtained at the boundary of its domain. Uniqueness of solutions to Poisson's equation is finally proved through the observation that the difference of two solutions equals a harmonic function.

2 The mean-value property

The mean-value property means that the value of a function u at a point x equals to the mean value of u in a sphere centered at x. We will show that harmonic functions have this property.

If we take the function $y = ax + b$ as an example in one dimension, see figure 1, we see that $y'' = 0$ so y is by definition harmonic. Since the domain is one dimensional a "sphere" around a point x_0 is just two points at an equally long distance from x_0 on the x-axis. Let h be the distance between x_0 and one of these points. If the mean value property holds for y the following equality should be true,

$$y(x) = \frac{y(x+h) - y(x-h)}{2},$$

which easily can be verified.

The mean-value property holds for balls as well, so that the value of a funcion u at a point x equals to the mean value of u in a ball centered at x. This would in our example mean that

$$\frac{1}{2h} \int_{x-h}^{x+h} y(z) dz = y(x),$$

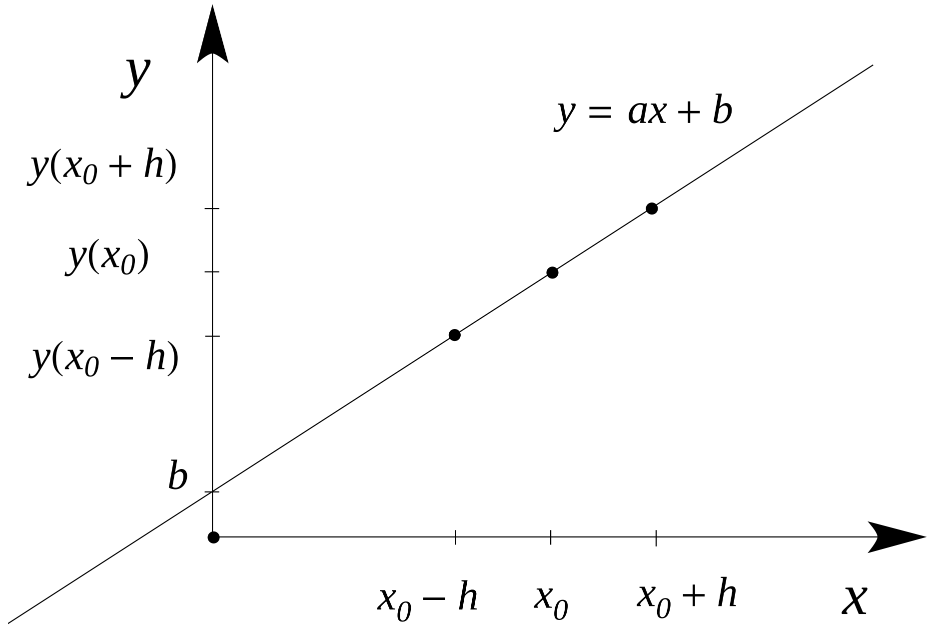

Figure 1: The very harmonic function y.

which also can be verified.

Theorem (The mean-value property)**.** *Let M be an open set (see Definition 3.1) in $\mathbb{R}^n$ and let u be a harmonic function in M. Also let $V(r)$ be the volume of a ball with radius r, $A(r)$ the area of a sphere with radius r and $B(x,r)$ a ball with center in point x and radius r. The boundary of $B(x,r)$, $\partial B(x,r)$, is the sphere of radius r centered at x. Then*[1]

$$u(x) = \frac{1}{A(r)} \int_{\partial B(x,r)} u(y)\,\mathrm{d}S(y) = \frac{1}{V(r)} \int_{B(x,r)} u(y)\,\mathrm{d}y.$$

Proof. The proof is divided into two parts where the property is shown for spheres in step one and balls in step two.

1. Keep earlier notations and let ϕ be the mean value of u on a sphere of radius r centered

[1] $\mathrm{d}S$ means the surface measure and $\mathrm{d}y$ means the volume measure.

around x,

$$\phi(r) = \frac{1}{A(r)} \int_{\partial B(x,r)} u(y)\, \mathrm{d}S(y).$$

We want to show that $\phi(r) = u(x)$. This is done by first showing that $\phi'(r) = 0$ and then using that $\lim_{t\to 0} \phi(t) = u(x)$. To do this we take advantage of u being harmonic and using one of Green's formulas, for which a new definition is needed.

Definition 2.1. *Let ∇ denote an operator with*

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right)$$

for a function u of n variables.

We start by differentiating ϕ but in order to do so we do not want r in the limits of the integral, and we also have that y depends on r. We therefore do the substitution $y = x + rz$ for a vector z and integrate over the unit sphere[2], see figure 2. We can note that z varies in the same extent as y which also makes u a function of z.

The mean value of u over these spheres will be the same. Observe that the new model has a new area which we have to divide by to get the mean value.

$$\frac{1}{A(r)} \int_{\partial B(x,r)} u(y)\, \mathrm{d}S(y) = \frac{1}{A(1)} \int_{\partial B(0,1)} u(x + rz)\, \mathrm{d}S(z).$$

We can differentiate ϕ under the integral sign [2] and according to the chain rule [2] we can differentiate $u(x + rz)$ with r as variable as[3]

$$\frac{d}{dr} u(x + rz) = \left(\frac{\partial u}{\partial (x + rz_1)}, \frac{\partial u}{\partial (x + rz_2)}, \dots, \frac{\partial u}{\partial (x + rz_n)} \right) \cdot (z_1, z_2, \dots, z_n) = \nabla u(x + rz) \cdot z,$$

[2]The sphere with radius one.
[3]See Definition 2.1 for ∇.

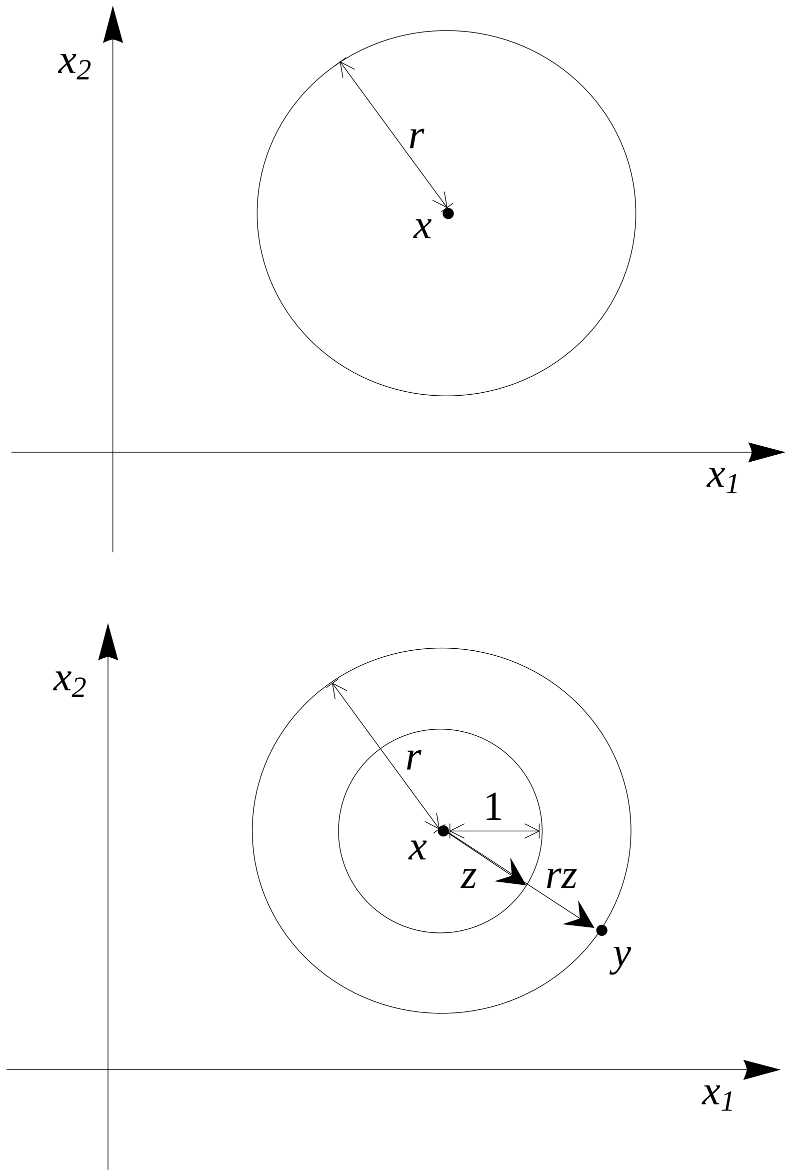

Figure 2: Geometry of the variable substitution

i.e. we have

$$\phi'(r) = \frac{1}{A(1)} \int_{\partial B(0,1)} \nabla u(x + rz) \cdot z \, \mathrm{d}S(z).$$

Returning to our original ball $B(x, r)$ gives, since $y = x + rz$,

$$\phi'(r) = \frac{1}{A(r)} \int_{\partial B(x,r)} \nabla u(y) \cdot \frac{y - x}{r} \, \mathrm{d}S(y).$$

The outer unit normal vector ν to B is, however, z, as seen in figure 2. The scalar product between ∇u and the normal vector ν to the domain equals the normal derivative (see [3]),

$$\nabla u \cdot \nu = \frac{\partial u}{\partial \nu}.$$

Thus,

$$\phi'(r) = \frac{1}{A(r)} \int_{\partial B(x,r)} \frac{\partial u(y)}{\partial \nu} \, \mathrm{d}S(y). \tag{1}$$

According to Green's formula, see [3], the following applies for M,

$$\int_{\partial M} \frac{\partial u(y)}{\partial \nu} dS(y) = \int_M \Delta u(y) dy.$$

Inserting this in (1) gives:

$$\phi'(r) = \frac{1}{A(r)} \int_{B(x,r)} \Delta u(y) dy.$$

However, $\Delta u = 0$ in M by assumption so

$$\phi'(r) = \frac{1}{A(r)} \int_{B(x,r)} \Delta u(y) dy = 0,$$

And as a result $\phi(r)$ is constant! This implies that the mean value of the ball is the same no matter what the radius. We can therefore imagine a new radius t which approaches 0. But the mean value of a continuous function over a sphere with center in x will approach the value at x when the radius goes towards 0, i.e.

$$\phi(r) = \lim_{t\to 0} \phi(t) = \lim_{t\to 0} \frac{1}{A(t)} \int_{\partial B(x,t)} u(y) dS(y) = u(x).$$

We have proved that the mean value of a harmonic function over every sphere around any given point equals the value at that point. What remains to be shown is that the same principle applies for balls,

$$u(x) = \frac{1}{V(r)} \int_{B(x,r)} u(y) dy.$$

2. A ball with radius r can be seen as the sum of spheres with radius r and less. This means that if we let s be a new radius that grows from 0 to r and sum the value of our function u over all these spheres we will have a sum of the value of u over a whole ball with radius r,

$$\int_{B(x,r)} u(y) dy = \int_0^r \left(\int_{\partial B(x,s)} u(y) dS(y) \right) ds. \tag{2}$$

But we had

$$u(x) = \phi(s) = \frac{1}{A(s)} \int_{\partial B(x,s)} u(y) dS(y),$$
$$\int_{\partial B(x,s)} u(y) dS(y) = u(x) A(s).$$

Inserting this into (2) gives

$$\int_{B(x,r)} u(y) dy = \int_0^r u(x) A(s) ds,$$

which since $u(x)$ does not depend on s equals

$$\int_{B(x,r)} u(y) dy = \int_0^r u(x) A(s) ds = u(x) \int_0^r A(s) ds.$$

But the primitive function of $A(s)$ is $V(s)$ so we have

$$\int_{B(x,r)} u(y)dy = u(x) \int_0^r A(s)ds = u(x)(V(r) - V(0)) = u(x)(V(r)).$$

Dividing by $V(r)$ finally gives

$$u(x) = \frac{1}{V(r)} \int_{B(x,r)} u(y)dy.$$

And thus proving the fact that the mean-value property holds for harmonic functions is done. □

Remark. *It can be shown that the converse is true as well, that functions having the mean value property are harmonic (see [3]).*

3 The maximum/minimum principle

For a harmonic function, the maximum and minimum of the function will be attained on the boundary. This is called the maximum/minimum principle. Also, if the set M is connected and the function u has an interior max/min, then u is constant.

What we want to show is that this principle applies for harmonic functions.

Definition 3.1. *Let M be a set in $\mathbb{R}^n$. M is called* open *if one for every element x in M can make a ball around it which lies completely in M.*

This implies that an open set M has no points on its boundary, since you can not make a ball around a point on the boundary without getting points outside of the set. No matter how close an element is to the boundary, however, a ball can always be made around it with all its points inside the set.

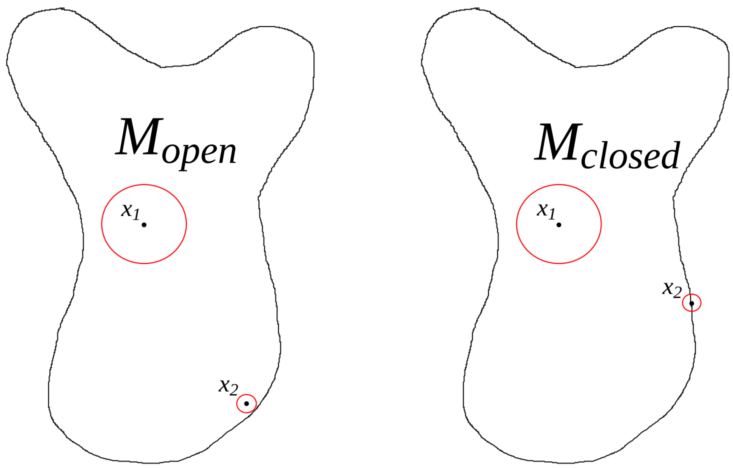

Figure 3: M_{open} has no points on the boundary while M_{closed} has got plenty.

Definition 3.2. *Let M be a set in $\mathbb{R}^n$. M is called* closed *if the complement to M is open.*

This implies that a closed set M has points on its boundary, since the complement to M, which is everything in $\mathbb{R}^n$ except the elements of M, is open, i.e. the boundary of M cannot be a part of the complement and is therefore part of M.

Theorem (The maximum/minimum principle)**.** *Let M be an open set in $\mathbb{R}^n$ and let $\overline{M}$ be the closure[4] of M. Also let u be a harmonic function in $\overline{M}$. Then*

1. *The maximum is attained on the boundary,*

$$\max_{\overline{M}} u = \max_{\partial M} u.$$

2. *Assume M is connected and x_0 is an inner point with $u(x_0) = \max_{\overline{M}} u$, then u is constant within $\overline{M}$.*

The same applies for the minimum value.

[4]The boundary included.

Proof. Since property 2 implies property 1 only property 2 will be proved. We also only show the maximum principle but a similar argument proves the minimum principle, $\min_{\overline{M}} u = \min_{\partial M} u$.

Let $m = \max_{\overline{M}} u = u(x_0)$ and imagine a ball, $B(x_0, r)$, with center in the point x_0 and radius r. Let $r < \text{dist}(x_0, \partial M)$ where $\text{dist}(x_0, \partial M)$ denotes the shortest distance from x_0 to the boundary, see figure 4. This means that $B(x_0, r)$ is included in M so u is defined there.

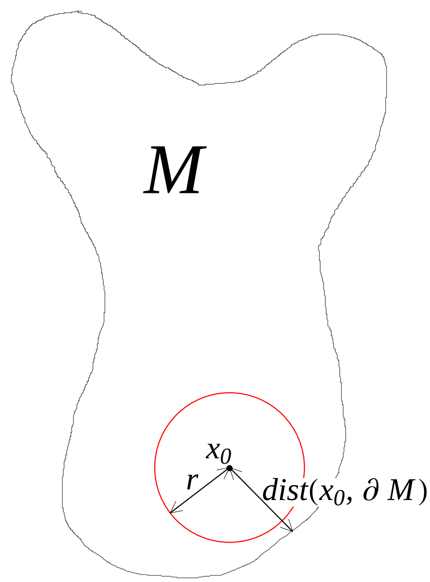

Figure 4: The radius has to be smaller than the distance to the boundary.

According to the mean-value property we have

$$m = u(x_0) = \frac{1}{V(r)} \int_{B(x_0,r)} u(y)\,\mathrm{d}y. \tag{3}$$

Since we have assumed m is a maximum value there can not exist any element in M for which $u > m$. Also (3) implies that there can not be any element inside $B = B(x_0, r)$ for which $u < m$ either. This means that $u = m$ for every element in B. We can in turn do the same thing with all elements in B and show that the value of u of all the other elements in all balls equal m as well, and this way get a chain of balls where $u = m$, see figure 5. What we thus have left to show is that every element in M can be reached by this chain and that the elements on the boundary also equal m.

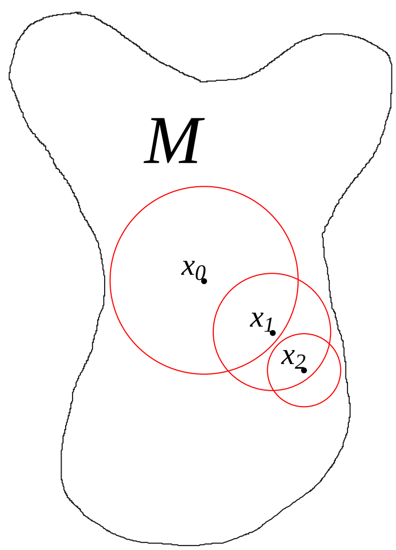

Figure 5: A chain of balls with the property $u(x) = m$ for every included element x.

We have however assumed that M is open so according to the definition we can imagine a ball around every element in M within which all points are elements in M. And since M is connected there must also be a path of elements that connects every element in M. Therefore we can reach every element in M with a chain of balls.

Since u is continuous up to the boundary[5] and we can choose elements infinitely close with the property $u = m$, we have that u must be equal to m on the boundary as well. $\square$

[5]This follows from the function being harmonic in $\overline{M}$.

Remark. *Property 1 will hold even if M is not connected because if a maximum/minimum value occurs for an inner value in a component the same value has to occur at the boundary of this component. However u may vary in the other components.*

4 Uniqueness

The purpose of this paper is to show that there exists only one solution to Poisson's equation. We do this by assuming the existence of two solutions and showing that these two must be identical.

Theorem (Uniqueness)**.** *Let M be an open set in $\mathbb{R}^n$. Then there exists at most one harmonic function u in M which is a solution to*

$$\begin{cases} \Delta u = f & \text{in } M, \\ u = g & \text{on } \partial M. \end{cases} \tag{4}$$

Proof. Assume that u and v are two solutions to (4) and let w be the difference between these two functions,

$$w = u - v. \tag{5}$$

From equation (4) and (5) we have

$$\begin{cases} \Delta w = \Delta(u - v) & \text{in } M \\ w = u - v = g - g = 0 & \text{on } \partial M, \end{cases}$$

and we also have $\Delta(u - v) = \Delta u - \Delta v$,

$$\begin{cases} \Delta w = \Delta u - \Delta v = f - f = 0 & \text{in } M \\ w = 0 & \text{on } \partial M, \end{cases}$$

However if $\Delta w = 0$ in M, then w is harmonic and the maximum/minimum value principle applies for w. But if w has both its maximum and minimum value on the boundary and $w = 0$ on ∂M then w must equal to 0 in whole $\overline{M}$. This means that $u(x) = v(x)$ for all x in $\overline{M}$ and we have proven that there can at most exist one solution to Poisson's equation. □

5 Conclusion

Harmonic functions have some very interesting properties such as the ones shown in this paper and one can derive uniqueness in solutions to Poisson's equation through these. It can also be shown that there always exists a solution [3] so we actually have that the Poisson equation always has one and one only solution.

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